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# WOOHOO! COOL MATH!

## CURIOUS MATHEMATICS FOR FUN AND JOY

## JANUARY 2016

**PROMOTIONAL CORNER:** *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep, joyous, and real mathematical doing I'll happily mention it here.*

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*Uplifting Mathematics for All*

The Global Math Project is an invitation to students, teachers, and communities everywhere to actively foster their sense of wonder and to enjoy truly uplifting mathematics. Math is a human endeavor. It's about thinking creatively, exploring patterns, explaining structure, and solving real problems. Lookout for our first GLOBAL MATH WEEK! See www.theglobalmathproject.org for details and how to support. (See also https://www.theglobalmathproject.org/newsletter/december2015/ for cool math in our first update report.)

**OPENING PUZZLE:** I have 100 pennies. In how many different ways can I split them into three piles? (Consider the order of piles immaterial. For example, a split into piles of 1 penny, 10 pennies, and 89 pennies is considered the same as split into piles of 10 pennies, 89 pennies, and 1 penny.)

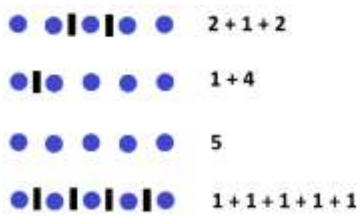
## THE PARTITION NUMBERS

A *partition* of a positive integer $n$ is a split of that number into a sum of one or more counting numbers. For example, there are four ordered partitions of the number three: $3$ and $2+1$ and $1+2$ and $1+1+1$.

If order is considered immaterial, then we have just three unordered partitions of three: $3$ and $2+1$ and $1+1+1$.

Counting the number of ordered partitions of a positive integer $n$ is quite manageable: there are $2^{n-1}$ of them.

**Reason:** Imagine $n$ dots in a row. Each ordered partition of $n$ corresponds to a placement of some or no "breaks" between dots.




There are $n-1$ spaces between dots and with each space we have the choice to break or not to break. That gives $2^{n-1}$ options in all.

Counting the number of unordered partitions of an integer $n$, however, is a different matter!

Let $P(n)$ denote the number of unordered partitions of $n$. The first few values for $P(n)$ (beyond $P(1)=1$) are prime numbers

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(n)$ | 1 | 2 | 3 | 5 | 7 | 11 |

.

But that is just coincidental. Next we have $P(7)=15$, $P(8)=22$, $P(9)=30$, and $P(10)=42$.

No one on this planet currently knows an explicit formula for these numbers. Understanding unordered partitions remains a topic of interest today.

Indian mathematician Srinivasa Ramanujan (1887 - 1920) guessed that $P(n)$ is well approximated by the surprising formula

$$\frac{1}{4\pi\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$$

for $n$ large. This was later proved correct by G.H. Hardy and Ramanujan together.

**Question:** What is the approximate value of $P(100)$?

Because of their notoriety the values of $P(n)$ are known as <u>the</u> partition numbers.

**CHALENGE:** Here are all of <u>the</u> partitions of the number five:

**5**
**4+1**
**3+2**
**3+1+1**
**2+2+1**
**2+1+1+1**
**1+1+1+1+1**

The number $1$ is listed $12$ times.

Prove: *The number of* $1$*s that appear among all the partitions of an integer* $n$ *is* $1+P(1)+P(2)+\cdots+P(n-1)$.

In the partitions of five, 5 appears in 1 partition, 4 appears in 1 partition, 3 appears in 2 of the partitions, 2 appears in 3 of them, and 1 appears in 5 of them. We have $1+1+2+3+5=12$ again. Why is this not a coincidence?

## PARTITIONS INTO A FIXED NUMBER OF PARTS

Let $P_k(n)$ be the number of (unordered) partitions of $n$ into precisely $k$ parts. For example, $P_3(5)=2$ as there are precisely two ways to write $5$ as a sum of three terms: $3+1+1$ and $2+2+1$.

The opening puzzler is asking for the value of $P_3(100)$.

Although there is no known formula for $P(n)$, might there be formulas for some, or all, of the $P_k(n)$?

**Formula for $P_1(n)$:**

We have $P_1(n)=1$ for all $n$.

**Formula for $P_2(n)$:**

We see that $P_2(100)=50$ (from $1+99$, $2+98$, …, $50+50$) and, in general, $P_2(n)=\frac{n}{2}$ if $n$ is even.

We also see that $P_2(101)=50$ (from $1+100$, $2+99$, …, $50+51$) and, in general, $P(n)=\frac{n-1}{2}$ if $n$ is odd.

We can combine these cases with a single formula.

$$P_2(n)=\left\lfloor \frac{n}{2} \right\rfloor$$

where the square-ish brackets mean to round down to the nearest integer.

**Formula for $P_3(n)$:**

So far, so good. But matters are now much trickier for partitions into three parts. Do you see any structure or pattern to the first twenty values of $P_3(n)$?

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P_3(n)$ | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 12 | 14 | 16 | 19 | 21 | 24 | 27 | 30 | 33 |

(I didn't when I first considered these numbers and computed the above list by hand.)

## CRACKING THE $P_3(n)$ FORMULA

$P_3(n)$ is the number of ways to split a set of $n$ pennies into three piles. Thinking of matters this way, it seems that there should be some fundamental connection between the vales of $P_3(n)$ and $P_3(n+3)$. After all, adding one penny to each pile in splitting $n$ pennies into three piles gives a three-pile splitting of $n+3$ .

**Lemma:** $P_3(2m)=P_3(2m-3)+m-1$ *for* $m\ge 2$.

**Proof**: There are $P_3(2m-3)$ ways to split $2m-3$ pennies into three piles. Add one penny to each pile and we now have all the ways to split $2m$ pennies into three piles with at least $2$ pennies per pile. But we are "missing" all the ways to split $2m$ pennies into three piles with at least one pile of $1$. Here are the possibilities we are missing:

$$2m-2,1,1$$
$$2m-3,2,1$$
$$2m-4,3,1$$
$$\ldots$$
$$m,m-1,1$$

There are $m-1$ of these.

**Lemma:** $P_3(2m-3) = P_3(2m-6) + m - 2$ *for* $m \geq 4$.

**Proof:** There are $P_3(2m-6)$ ways to split $2m-6$ pennies into three piles. Add one penny to each pile and we now have all the ways to split $2m-3$ pennies into three piles with at least $2$ pennies per pile. But we are "missing" all the ways to split $2m-3$ pennies into three piles with at least one pile of $1$. Here are the possibilities we are missing:

$$2m-5,1,1$$
$$2m-6,2,1$$
$$2m-7,3,1$$
$$\ldots$$
$$m-2,m-2,1$$

There are $m-2$ of these.

With these lemmas we can focus on either the even indexed or the odd indexed terms of the sequence of values $P_3(n)$. If we have a formula for $P_3(n)$ with $n$ even, say, then we can use either lemma to get a formula for $P_3(n)$ with $n$ odd.

Let's so focus on the even-indexed terms.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P_3(n)$ | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 12 | 14 | 16 | 19 | 21 | 24 | 27 | 30 | 33 |

Combining the two lemmas we get a formula for how terms six places apart differ.

$$\begin{aligned} P_3(2m) &= P_3(2m-3) + m - 1 \\ &= P_3(2m-6) + m - 1 + m - 2 \\ &= P_3(2m-6) + 2m - 3. \end{aligned}$$

Write this as

$$P_3(2m) = 2m - 3 + P_3(2m-6).$$

Now every even number can be written in the form $6a + r$ for some integer $a$ and with $r = 0$, $2$, or $4$. We see:

$$\begin{aligned} &P_3(6a+r) \\ &\quad = 6a + r - 3 \\ &\qquad + P_3(6(a-1)+r) \\ &\quad = 6a + r - 3 \\ &\qquad + 6(a-1) + r - 3 \\ &\qquad + P_3(6(a-2)+r) \\ &\quad = 6a + r - 3 \\ &\qquad + 6(a-1) + r - 3 \\ &\qquad + 6(a-2) + r - 3 \\ &\qquad + P_3(6(a-3)+r) \\ &\quad = 6a + r - 3 \\ &\qquad + 6(a-1) + r - 3 \\ &\qquad + 6(a-2) + r - 3 \\ &\qquad + \cdots \\ &\qquad + 6 \cdot 1 + r - 3 \\ &\qquad + P_3(r) \end{aligned}$$

Using the fact that

$$1 + 2 + \cdots + (a-1) + a = \frac{a(a+1)}{2}$$

we see

$$\begin{aligned} P_3(6a+r) &= 3a(a+1) + ar - 3a + P_3(r) \\ &= 3a^2 + ar + P_3(r) \end{aligned}.$$

Rewriting this so that the expression $6a + r$ is explicit, we get

$$P_3(6a+r) = \frac{(6a+r)^2}{12} + \left(P_3(r) - \frac{r^2}{12}\right).$$

Now $P_3(6a+r)$ is an integer (it is counting something), but $\dfrac{(6a+r)^2}{12}$ need not be.

Thus the term $P_3(r)-\frac{r^2}{12}$ must be a fractional amount that brings $\frac{(6a+r)^2}{12}$ to an integral value.

We can check:

When $r=0$, $P_3(r)-\frac{r^2}{12}=0$;

when $r=2$, $P_3(r)-\frac{r^2}{12}=0-\frac{1}{3}=-\frac{1}{3}$ ;

and

when $r=4$, $P_3(r)-\frac{r^2}{12}=1-\frac{4}{3}=-\frac{1}{3}$.

So $P_3(r)-\frac{r^2}{12}$ is always less than one half.

Consequently $\frac{(6a+r)^2}{12}+\left(P_3(r)-\frac{r^2}{12}\right)$ is $\frac{(6a+r)^2}{12}$ rounded to the nearest integer.

We have:

$$P_3(2m)=\left\langle\frac{(2m)^2}{12}\right\rangle$$

where the angle brackets mean to round to the nearest integer.

[Explicitly: $P_3(2m)=\frac{(2m)^2}{12}$ if $2m$ is a multiple of $6$ and $P_3(2m)=\frac{(2m)^2}{6}-\frac{1}{3}$ otherwise.]

Now to obtain a formula for the odd terms.

Using the first lemma,

$$\begin{aligned}P_3(2m-3) &= P_3(2m)-m+1\\ &=\frac{(2m)^2}{12}-m+1+\left(0 \text{ or } -\frac{1}{3}\right)\\ &=\frac{(2m-3)^2}{12}+\frac{1}{4}+\left(0 \text{ or } -\frac{1}{3}\right)\\ &=\frac{(2m-3)^2}{12}+\left(\frac{1}{4} \text{ or } -\frac{1}{12}\right).\end{aligned}$$

We know the answer must be an integer, so the additional fractional amounts must be rounding the quantity $\frac{(2m-3)^2}{12}$ to the nearest integer.

We have:

$$p_3(2m-3)=\left\langle\frac{(2m-3)^2}{12}\right\rangle.$$

Overall, we have:

*The number of ways to partition the positive integer $n$ into three parts is*

$$P_3(n)=\left\langle\frac{n^2}{12}\right\rangle.$$

There are thus 833 ways to split 100 pennies into three piles.

So we have, so far:

$$\begin{aligned}P(n) &= P_1(n)+P_2(n)+P_3(n)+P_4(n)+\cdots\\ &=1+\left\lfloor\frac{n}{2}\right\rfloor+\left\langle\frac{n^2}{12}\right\rangle+P_4(n)+\cdots\end{aligned}$$

But that's as far as I am willing to go!

**CHALLNGE:** Let $T(n)$ be the number of different triangles one can make using precisely $n$ toothpicks. (That is, $T(n)$ is the count of incongruent triangles of perimeter $n$ with integer side lengths.)

| n: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T(n): | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 1 | 3 | 2 | 4 | 3 | 5 | 4 | 7 | 5 | 8 | 7 | 10 | 8 |

For example, one can make four integer triangles of perimeter 11. (And surprisingly only three triangles if you add one more toothpick to the perimeter!)

Prove that $T(n) = \left\langle \frac{n^2}{48} \right\rangle$.

**Extra**: Let $S(n)$ be the count of incongruent scalene triangles of perimeter $n$ with integer side lengths. Find a formula for $S(n)$.

**RESEARCH CORNER**

Find a formula for each of $P_4(n)$, $P_5(n)$, $P_6(n)$, ….

Find an explicit formula for $P(n)$.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