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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JULY 2014

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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For some reason in my high-school teaching career I was assigned to teach multiple sections of geometry each and every year. (Geometry. Geometry. Always geometry.) Fortunately I was given the freedom to teach the subject in my own Tanton-way. And now that way is a course with THE GREAT COURSES company! Check out http://www.thegreatcourses.com/tgc/courses/course_detail.aspx?cid=1033

PUZZLER:
a) Give an example of one hundred consecutive integers with each a composite number.

b) Show that there exist one-hundred consecutive integers each divisible by at least one-hundred distinct primes.

WARNING: Some parts of this essay assume familiarity with infinite sums and results from calculus.

AN UNUSUAL QUESTION

Here are the prime factorizations of the first ten positive integers:

$$1 = 2^0 3^0 5^0 7^0 11^0 13^0 \cdots$$
$$2 = 2^1 3^0 5^0 7^0 11^0 13^0 \cdots$$
$$3 = 2^0 3^1 5^0 7^0 11^0 13^0 \cdots$$
$$4 = 2^2 3^0 5^0 7^0 11^0 13^0 \cdots$$
$$5 = 2^0 3^0 5^1 7^0 11^0 13^0 \cdots$$
$$6 = 2^1 3^1 5^0 7^0 11^0 13^0 \cdots$$
$$7 = 2^0 3^0 5^0 7^1 11^0 13^0 \cdots$$
$$8 = 2^3 3^0 5^0 7^0 11^0 13^0 \cdots$$
$$9 = 2^0 3^2 5^0 7^0 11^0 13^0 \cdots$$
$$10 = 2^1 3^0 5^1 7^0 11^0 13^0 \cdots$$

The average exponent of the prime 2 is:

$$\frac{0+1+0+2+0+1+0+3+0+1}{10} = 0.8 .$$

The average exponent of the prime 3 is 0.4, of 5 its 0.2, of 7 its 0.1, and its zero thereafter. So the "average factorization" for these ten integers is:

$$2^{0.8} 3^{0.4} 5^{0.2} 7^{0.1} 11^0 13^0 \cdots$$

(This equals about 4.529 if that means anything.)

Exercise: What is the average prime factorization of the first twenty positive integers? The first one hundred?

Here's our question:

What is the average prime factorization of all the positive integers?

COUNTS OF SQUARE FACTORS

Consider the set of square numbers: $S = \{1, 4, 9, 16, 25, ...\}$.

Side Comment: I like the square numbers because the infinite sum of their reciprocals, $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$, equals $\frac{\pi^2}{6}$. I find this connection between squares and circles delightfully quirky. (See www.jamestanton.com/?p=720 for a proof of this astounding result.)

Let $Sq(n)$ denote the number of square factors n possesses. For example, $Sq(18) = 2$ because 18 has just the squares 1 and 9 as factors, $Sq(7) = 1$, and $Sq(100) = 4$.

The following table shows the square number factors of the first 20 integers. In it we have that $Sq(n)$ is the number of Xs in the nth column. (The 18th column contains two X s and $Sq(18) = 2$.)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
4				X				X				X				X				X
9									X									X		
16																X				
25																				
⋮																				

Summing over all twenty columns and dividing by 20 gives the average count of square factors each of the numbers 1 through 20 possesses. This is the total number of X s in the table divided by 20:

$$Ave\,Square(20) = \frac{Sq(1)+Sq(2)+\cdots+Sq(20)}{20}$$
$$= \frac{28}{20} = 1.4$$

.

We could count the total number of X s in the table summing by rows of X s instead. This would give us a second way to compute the average count of square factors.

How many X s are there in a particular row of the table?

There are two multiples of nine—1×9 and 2×9—each less than or equal to 20 and so there are two X s on the row for 9. In general, for a table with n columns, the number of X s in the row for the number a is the largest k so that ka is less than or equal to n: $ka \le n$. That is, k is the largest integer less than or equal to $\frac{n}{a}$. This is $\left\lfloor \frac{n}{a} \right\rfloor$, the quantity $\frac{n}{a}$ rounded down to the nearest integer. ($\left\lfloor \frac{10}{3} \right\rfloor = 3$, $\left\lfloor \frac{10}{4} \right\rfloor = 2$, and $\left\lfloor \frac{10}{5} \right\rfloor = 2$, for example.)

The total number of X s in the table of twenty columns for the first 20 integers is:

$$\left\lfloor \frac{20}{1} \right\rfloor + \left\lfloor \frac{20}{4} \right\rfloor + \left\lfloor \frac{20}{9} \right\rfloor + \left\lfloor \frac{20}{16} \right\rfloor + \left\lfloor \frac{20}{25} \right\rfloor + \left\lfloor \frac{20}{36} \right\rfloor + \cdots$$

$$= \lfloor 20 \rfloor + \lfloor 5 \rfloor + \lfloor 2.22.. \rfloor + \lfloor 1.25 \rfloor + \lfloor 0.8 \rfloor + \lfloor 0.55... \rfloor + \cdots$$
$$= 20 + 5 + 2 + 1 + 0 + 0 + \cdots$$
$$= 28$$

just as we have seen before.

But this second approach gives us a general formula for the average count of square factors of the first n integers:

$$Ave\,Square(n) = \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{9} \right\rfloor + \left\lfloor \frac{n}{16} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \cdots \right)$$

(Notice that only the first few terms in this sum are non-zero: $\left\lfloor \frac{n}{a} \right\rfloor = 0$ as soon as $a > n$.)

Question: $\left\lfloor \frac{n}{a} \right\rfloor$ differs from $\frac{n}{a}$ only by a fraction smaller than 1. Is it reasonable to write …

$$Ave\,Square(n)$$
$$= \frac{1}{n}\left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{9} \right\rfloor + \left\lfloor \frac{n}{16} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \cdots \right)$$
$$\approx \frac{1}{n}\left(\frac{n}{1} + \frac{n}{4} + \frac{n}{9} + \frac{n}{16} + \frac{n}{25} + \cdots \right)$$
$$= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \frac{1}{36} + \cdots$$
$$= \frac{\pi^2}{6}\ ?$$

Could it be that, on average, a number possesses $\frac{\pi^2}{6}$ square factors?

Exercise: Did you compute the average count of square factors of the first 100 integers? Is that answer close to $\frac{\pi^2}{6} \approx 1.645$?

ESTIMATING THE ERROR:

Write $\left\{ \frac{n}{a} \right\}$ for the fractional part of $\frac{n}{a}$.

(For example, $\left\{ \frac{10}{3} \right\} = 0.33\ldots$, $\left\{ \frac{10}{4} \right\} = 0.5$ and $\left\{ \frac{10}{5} \right\} = 0$.) We have:

$$\frac{n}{a} = \left\lfloor \frac{n}{a} \right\rfloor + \left\{ \frac{n}{a} \right\}.$$

Let k be the number of non-zero terms in the sum:

$$Ave\,Square(n) = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + \left\lfloor\frac{n}{9}\right\rfloor + \left\lfloor\frac{n}{16}\right\rfloor + \left\lfloor\frac{n}{25}\right\rfloor + \cdots\right)$$

This k is given by the largest integer satisfying $k^2 \le n$. That is, $k = \left\lfloor \sqrt{n} \right\rfloor$.

So we can write:

$$Ave\,Square(n) = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + \left\lfloor\frac{n}{9}\right\rfloor + \cdots + \left\lfloor\frac{n}{k^2}\right\rfloor\right)$$

$$= \frac{1}{n}\left(\left(\frac{n}{1} - \left\{\frac{n}{1}\right\}\right) + \left(\frac{n}{4} - \left\{\frac{n}{4}\right\}\right) + \left(\frac{n}{9} - \left\{\frac{n}{9}\right\}\right) + \cdots + \left(\frac{n}{k^2} - \left\{\frac{n}{k^2}\right\}\right)\right)$$

$$= \frac{1}{n}\left(\frac{n}{1} + \frac{n}{4} + \frac{n}{9} + \cdots + \frac{n}{k^2}\right) - \frac{1}{n}\left(k \text{ fractions between } 0 \text{ and } 1\right)$$

$$= \left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}\right) - Error(n)$$

k is the largest integer just under, or equal to $\sqrt{n}$ (and so k grows as n is chosen to be larger and larger) and

$$Error(n) = \frac{\text{the } k \text{ fractions between } 0 \text{ and } 1}{n}$$

is the "error term" quantifying how $Ave\,Square(n)$ differs from the sum of reciprocals $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}$.

Since each fraction is between 0 and 1 we have:

$$0 \le Error(n) \le \frac{k}{n}.$$

Also, $\frac{k}{n} \le \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ which goes to zero as n grows. This means that the error term tends to zero. We have:

$$\lim_{n\to\infty} Ave\,Square(n) = \lim_{n\to\infty}\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}\right) - Error(n)$$

$$= \left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots\right) - 0$$

$$= \frac{\pi^2}{6}$$

We can interpret this as indeed saying:

On average, over <u>all</u> integers, each counting number possesses $\frac{\pi^2}{6}$ square factors.

IN GENERAL…

One can repeat the previous argument for any set of counting numbers:

$$S = \{a_1, a_2, a_3, \ldots\}.$$

We can say:

Let $S(n)$ be the number of entries of S that are $\le n$. If $\frac{S(n)}{n} \to 0$ as $n \to \infty$ then, on average, each integer possesses

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

factors from the set S.

Example: If S is the set of square numbers, then $S(n) = \lfloor \sqrt{n} \rfloor \le \sqrt{n}$, and $\frac{S(n)}{n} \le \frac{1}{\sqrt{n}} \to 0$ as n grows.

Example: If S is the set of fourth powers, then $S(n) = \lfloor \sqrt[4]{n} \rfloor$ and $\frac{S(n)}{n} \le \frac{1}{n^{\frac{3}{4}}} \to 0$ as $n \to \infty$. On average, each integer has $\frac{1}{1} + \frac{1}{16} + \frac{1}{81} + \frac{1}{625} + \cdots = \frac{\pi^4}{90}$ fourth-power factors.

Example: On average each integer has $\frac{1}{1} + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$ cube factors. No one on this planet currently knows the value of this sum of cube reciprocals.

Research Corner 1: For world fame, find some other means to compute the average count of cube factors of the integers and hence evaluate the sum:

$$\zeta(3) = \frac{1}{1} + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots.$$

Example: The set S could be a finite. For instance, if $S = \{3\}$, then for $n \ge 3$ we have $\frac{S(n)}{n} = \frac{1}{n} \to 0$ as n grows. We conclude that, on average, an integer possesses 3 as a factor $\frac{1}{3}$ of the time.

Question: How do you interpret our result for the set $S = \{2,3\}$?

Exercise: Show that, on average, an integer possesses two triangular numbers as factors. (The triangular numbers are the numbers $1,3,6,10,15,21,28,\ldots$.)

THE MOST AVERAGE PRIME FACTORIZATION:

Consider the set:

$$S = \{2,4,8,16,32,64,\cdots\},$$

the non-trivial powers of two.

For an integer n, let's find $S(n)$ for this set. This is the number of powers of two that are less than or equal to n. Finding this count is equivalent to finding the largest integer k satisfying:

$$2^k \le n.$$

That is, we seek the largest k less than or equal to $\log_2 n$. Thus $S(n) = \lfloor \log_2 n \rfloor$.

Now $\frac{S(n)}{n} = \frac{\lfloor \log_2(n) \rfloor}{n} \le \frac{\log_2 n}{n} \to 0$ as n grows. (Use L'Hopital's rule from calculus.)

Thus by our result we can say that, on average, each integer possesses:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

powers of two as factors.

The geometric series formula

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

shows that this sum is:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = \frac{1}{1-\frac{1}{2}} - 1 = \frac{2}{2-1} - 1 = 1$$

Thus, on average, each integer has one power of two as a factor.

But let's interpret matters this way:

If a number has prime factorization:

$$2^a 3^b 5^c 7^d \cdots$$

then it has a powers of two as factors. The average value of a must be 1.

In the same way, looking at the non-trivial powers of three $S = \{3, 9, 27, 81, \ldots\}$ shows that the average value of the exponent b must be:

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1}{1 - \frac{1}{3}} - 1$$
$$= \frac{3}{3-1} - 1$$
$$= \frac{1}{2}$$

The average value exponent c is:

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{1 - \frac{1}{4}} - 1$$
$$= \frac{4}{4-1} - 1$$
$$= \frac{1}{3}$$

And in general, the average exponent of a prime p is:

$$\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{1}{1 - \frac{1}{p}} - 1$$
$$= \frac{p}{p-1} - 1$$
$$= \frac{1}{p-1}$$

We have:

The most average prime factorization is:

$$2^1 3^{\frac{1}{2}} 5^{\frac{1}{4}} 7^{\frac{1}{6}} 11^{\frac{1}{10}} 13^{\frac{1}{12}} \cdots$$

Exercise: For each integer n consider the largest value k so that $k!$ divides n. Show that, on average, this value equals:

$$e - 1 = 1.718281828459045\ldots.$$

RESEARCH:

I've been playing with this little result for several years now. I would be very interested in any hearing any thoughts you might have on this topic.

Research Corner 2: Consider the exponent of the largest power of ten that divides each integer. Show that, on average, this exponent is $1/9$. (So the average power of ten factor of an integer is $10^{\frac{1}{9}}$? How does this statement gibe with the most average prime factorization?)

Research Corner 3: Show that, on average, a number has $\ln 2$ more odd factors than even factors. Use

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

(Since we are now allowing negative factors, one needs to refine – delicately – the analysis of the error term.)

Research Corner 4: We showed that if $\frac{S(n)}{n} \to 0$ as n grows, then the average count of factors from the set S is the sum of the reciprocals of the integers in S.

Is this the only possible condition that guarantees this result?
Is there a problem with our result if the sum of reciprocals happens to be infinite?

Research Corner 6: For each integer, look at its prime factorization $2^a 3^b 5^c 7^d 11^e 13^f \cdots$ and work out the product

$$\left\lfloor \frac{a}{2} \right\rfloor \times \left\lfloor \frac{b}{2} \right\rfloor \times \left\lfloor \frac{c}{2} \right\rfloor \times \cdots,$$

but only include the non-zero terms in this product.

Is the average value of this product $\frac{\pi^2}{6} - 1$?

The product $\left\lfloor \frac{a}{3} \right\rfloor \times \left\lfloor \frac{b}{3} \right\rfloor \times \left\lfloor \frac{c}{3} \right\rfloor \times \cdots$?

THE OPENING PUZZLER:

Consider the one-hundred consecutive numbers

$$101!+2$$
$$101!+3$$
$$\vdots$$
$$101!+101$$

The first number is divisible by 2, the second by 3, the third by 4, all the way up to the 100th divisible by 101. We have a list of one-hundred consecutive composite integers.

Unfortunately the only way I know to answer the second puzzler is to make use of a famous result from number theory.

Let m_1 be the product of the first one-hundred primes, m_2 the product of the next one-hundred primes, m_3 the product of the hundred after that, and so on.

By the Chinese Remainder Theorem there is sure to be an integer N that is simultaneously…

1 less than a multiple of m_1
2 less than a multiple of m_2
3 less than a multiple of m_3
$\vdots$
100 less than a multiple of m_{100}.

This means that each of the integers

$$N+1, N+2, N+3, \ldots, N+100$$

is divisible by at least one-hundred distinct primes.

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