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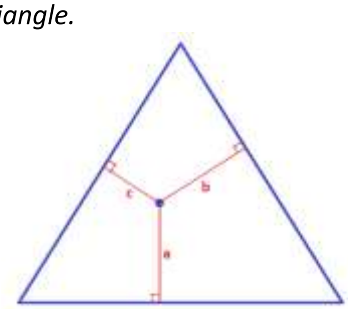
WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

February 2017

THIS MONTH'S PUZZLER

Last month we touched on Viviani's Theorem: *For any point inside an equilateral triangle, the sum of its distances from each of the three sides is constant. That constant is the height of the triangle.*



a + b + c = height

Is the converse true? If a triangle has the property that for any point inside the triangle the sum of its distances from each of the three sides of the triangle is a constant value, must the triangle be equilateral?

Further: A rectangle has this "constant sum" property too: For any point inside a rectangle, the sum of each of its distances from the sides of the figure is constant. What other quadrilaterals have this property? Is there a seven-sided figure with this property?

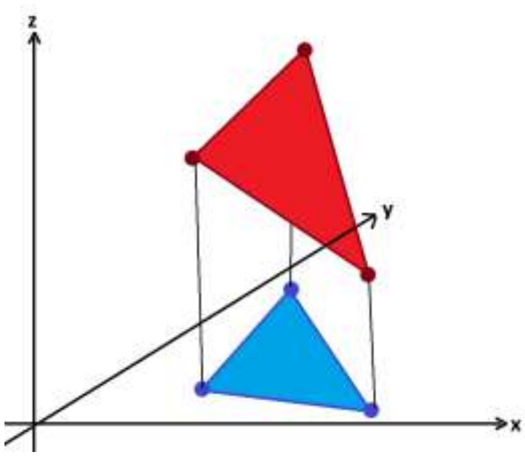
PLANES ABOVE TRIANGLES

Consider an arbitrary triangle in the plane.

For each point P inside the triangle, let $s(P)$ be the sum of each of its three distances from the sides of the triangle. So for each interior point of the triangle we get a number. Viviani says that this number is the same for each and every point inside an equilateral triangle. For arbitrary triangles, the numbers change from point to point.

Nonetheless, we have a function from the set of all points inside the triangle to the set of real numbers and we can draw a (three-dimensional) graph of this function: Place the triangle in the horizontal xy-plane, and above each point P inside the triangle plot a point in the z-direction $s(P)$ units high above it.

BOLD CLAIM: *The graph is sure to be a flat surface, that is, the graph is a section of a plane.*



This is quite the claim! As we shall see, the validity of this claim unlocks many secrets about Viviani's property for figures.

But to first prove the claim true, we must understand the algebra of lines, planes, and distances to lines.

THE EQUATION OF A PLANE

We know from high school that any equation of the form $y = Ax + B$ represents a line in the plane. When one later extends this work to three-dimensional coordinate geometry, we learn that the analogous equation $z = Ax + By + C$ corresponds to the equation of a plane.

Thus to prove our bold claim, we just need to show that for each point $P = (x, y)$ in the triangle, $s(P)$ is given by a formula of the form $Ax + By + C$.

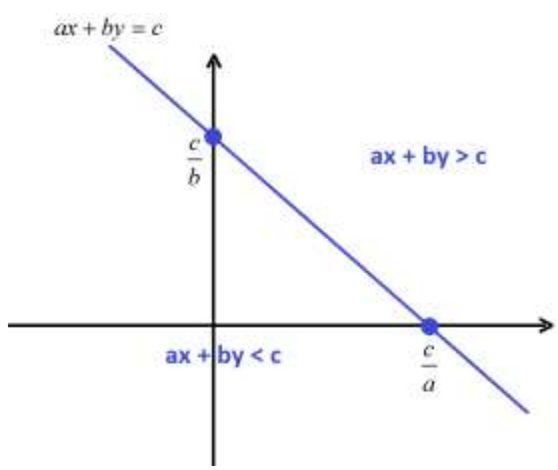
Each value $s(P)$ is a sum of three distances. So let's now work out distance formulas for a point from lines.

THE DISTANCE OF A POINT FROM A LINE

With some algebraic manipulation, every line in the plane can be expressed as an equation of the form

$$ax + by = c$$

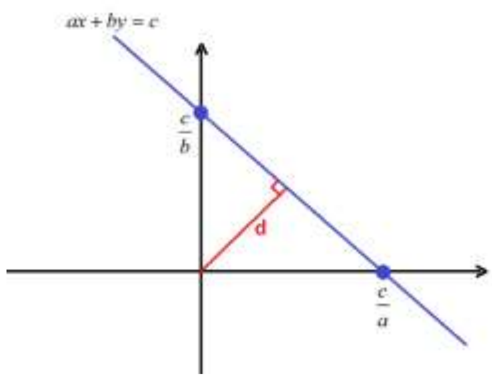
with c a non-negative number and a and b not both zero. (If c is zero, then we have a line through the origin.) This equation has the advantage that it encompasses vertical lines as well.



Each such line divides the plane into two regions: the region containing points arbitrarily far along the northeast diagonal corresponds to the set of all points (x, y) in the plane with $ax + by > c$. The other region, all points (x, y) with $ax + by < c$.

If each of a, b, and c is non-zero, then the given line misses the origin and encloses a right triangle in one of the quadrants. If a and b are both positive, then we see a right triangle in the first quadrant. If a is negative and b is positive, a right triangle in the second quadrant. And so on.

Let's compute the distance d of the origin from such a line.



We can do this by working out the area of the right triangle we see two different ways. Seeing the triangle as having horizontal base we get

$$\text{Area} = \frac{1}{2} \cdot \frac{c}{|a|} \cdot \frac{c}{|b|}.$$

Seeing the hypotenuse of the triangle as the base we get

$$\text{Area} = \frac{1}{2} \cdot \sqrt{\frac{c^2}{|a|^2} + \frac{c^2}{|b|^2}} \cdot d.$$

It follows that

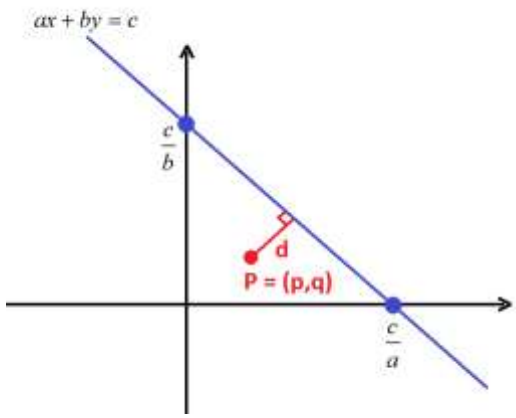
$$d = \frac{c}{|a| \cdot |b| \cdot \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{c}{\sqrt{a^2 + b^2}}.$$

The distance of the origin from the line $ax + by = c$, with $c \geq 0$, is

$$d = \frac{c}{\sqrt{a^2 + b^2}}.$$

This formula happens to be valid even if $c = 0$, that is, if the line passes through the origin. It is also valid if $a = 0$ or $b = 0$, that is, if the line is horizontal or vertical. Thus this formula is valid for all lines.

What is the distance of a general point $P = (p, q)$ from a line $ax + by = c$?



Rewrite the equation of the line as $a(x - p + p) + b(y - q + q) = c$, that is, as

$$a(x - p) + b(y - q) = c - ap - bq.$$

In this form of the equation, $x = p$ is "acting like zero" for the x-values, and $y = q$ is "acting like zero" for the y-values. That is, our point $P = (p, q)$ is acting like the origin for this line.

So the distance of P from our line $a(x - p) + b(y - q) = c - ap - bq$ matches the distance of the origin from the line $ax + bq = c - ap - bq$. And we know how to compute this distance.

If $c - ap - bq$ is a positive number, then the distance d we seek is $\frac{c - ap - bq}{\sqrt{a^2 + b^2}}$.

If, on the other hand, $c - ap - bq$ is negative, work with the equation $-ax - by = ap + bq - c$. Then the distance we seek is given by $\frac{ap + bq - c}{\sqrt{a^2 + b^2}}$.

Either way,

The distance of a point $P = (p, q)$ from a fixed line is given by a formula of the form

$$d = Ap + Bq + C$$

for some fixed values A, B, and C.

This is true even for vertical and horizontal lines.

With too many x s and y s in the mix we gave the point P coordinates p and q. If we now allow ourselves to write $P = (x, y)$, then we can say:

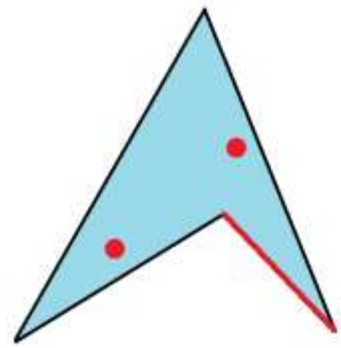
For a point $P = (x, y)$ inside a given triangle, the sum $s(P)$ of the three distances of P to each line defining a side of the triangle is given by a formula of the form $Ax + By + C$. Thus the graph of the function s above the triangle is a section of a plane.

We have proved the bold claim.

Actually more is true.

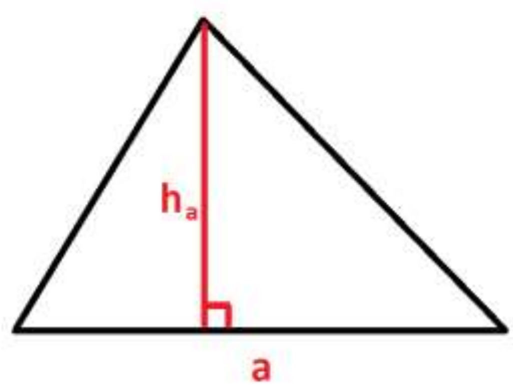
For a point P inside a given convex figure, as a sum of its distances to each of its sides, $s(P)$, is given by a formula of the form $Ax + By + C$. Thus the graph of s above the figure is again a section of a plane.

The result need not be true for concave figures as then two points inside the figure might sit on opposite sides of a line defining a given side. This leads to a sign change in our distance formula.



THE CONVERSE OF VIVIANI'S THEOREM

Suppose a triangle has a side of length a and altitude of length h_a as measured from that base. Then the area A of the triangle is given by $A = \frac{1}{2} a h_a$ showing that $a = \frac{2A}{h_a}$. Similarly, $b = \frac{2A}{h_b}$ and $c = \frac{2A}{h_c}$ for the remaining sides and altitudes of the triangle.

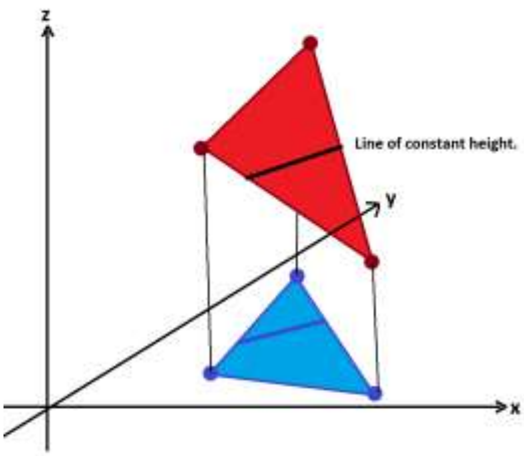


Now suppose this triangle has the property that the value of $s(P)$ the same for all points P inside the triangle. If P is the vertex of the triangle opposite the side of length a, then $s(P) = 0 + 0 + h_a = h_a$. So the constant value of $s(P)$ is h_a. But by the same token, if we chose a different vertex we would argue that the constant value of $s(P)$ is h_b or h_c. Thus we must have $h_a = h_b = h_c$, from which it follows that $a = b = c$ and the triangle is equilateral.

OTHER FIGURES

Three-dimensional graphs that are planes have the property that their contour lines, curves of constant height, are straight lines. (Unless the plane is horizontal, in which case there is one "line" of constant height, the whole plane itself.)

And recall we proved that the graph of s for any convex figure in the plane is planar.



Lines of constant height correspond to regions within the planar figure of constant s value, that is, points with the same sum of distances. Let's call these regions isosums.

For any convex figure in the plane with s adopting more than one value, the isosums divide the figure into parallel line segments. Each line segment spans across the figure.

This means that if, for a convex figure, P, Q, and R are three points with $s(P) = s(Q) = s(R)$, then either P, Q, and R lie on the same line segment, or the graph of s is a horizontal plane and the value of $s(P)$ is the same for all values in the figure.

Let's say a convex figure is "VP" if it has the Viviani property: *for each point P in the figure, the sum $s(P)$ of distances of P from each of the sides of the figure is constant.*

Equilateral triangles are VP.

We have:

If a convex figure has three non-collinear points with the same sum of distances, then the figure is VP.

If a convex figure has rotational symmetry, then the set of isosums has rotational symmetry as well. This means we can find three non-collinear points with the same sum of distances and so the figure must be VP.

All convex figures with rotational symmetry are VP.

Thus rectangles, parallelograms, and regular polygons are all VP.

Question: *What can we say about a convex figure with one line of reflection symmetry?*

RESEARCH CORNER

For each triangle, the section of the plane above it has equation $z = Ax + By + C$ for some values A, B, and C. What is the relationship between these values and the side lengths of the triangle, or with the area of the triangle, or with some other geometric feature of the triangle? (More precisely, what can we say about the normal vector to the plane, its length and its direction?)

What is the relationship between the area of the base triangle and the area of the triangular section of the plane above it? (Clearly these two areas match precisely if, and only if, the plane is horizontal and so the triangle is equilateral.)

Can we explore a VP property for unbounded regions of the plane?

What are the appropriate modifications to develop a meaningful analysis of the s function over concave regions? (Can we say that a rotationally symmetric star shape has VP if one allows measured distances to possibly be negative?)

Is there a VP property for points inside polyhedra? Higher dimensions?

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