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WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

FEBRUARY 2016

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep, joyous, and real mathematical doing I'll happily mention it here.*

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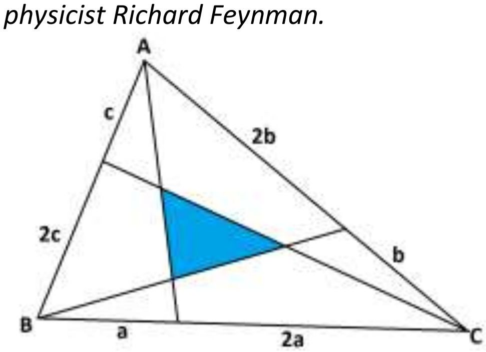
Fun math to be found at Marc Chamberland's *Tipping Point Math:* https://www.youtube.com/user/TippingPointMath Well worth checking out.

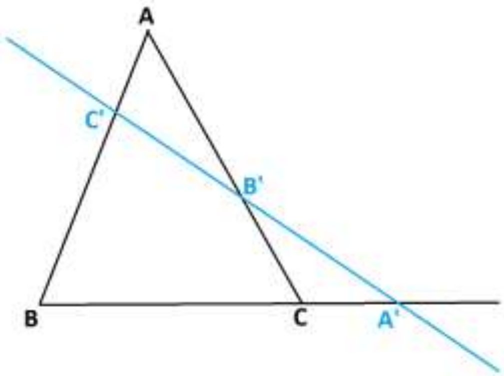
FEYNAM'S TRIANGLE PROBLEM:
This puzzle is said to be due to famous physicist Richard Feynman.



In ΔABC a line is drawn from each vertex hitting its opposite side at the one-third mark. What fraction of the whole area is the area of the small inner triangle formed by these three lines?

TWO OBSERVATIONS ABOUT TRIANGLES

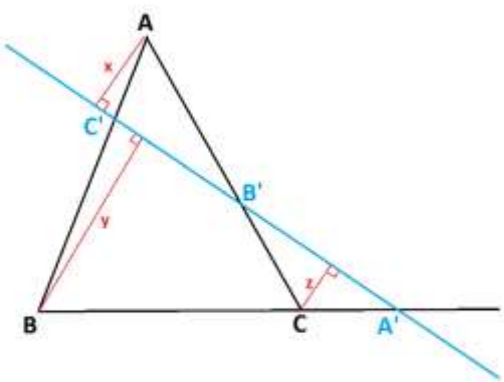
A line can intersect two sides of a triangle. But if we extend one side length of a triangle then it is possible for a line to intercept all three sides.



Greek scholar Menelaus of Alexandria (ca. 70 – 140 CE) showed that in such a situation, labelled as above, we must have:

$$\frac{AC'}{C'B}\times\frac{BA'}{A'C}\times\frac{CB'}{B'A}=1.$$

This is proved by drawing perpendiculars from the vertices of the triangle to the line and chasing through the similar triangles that result.

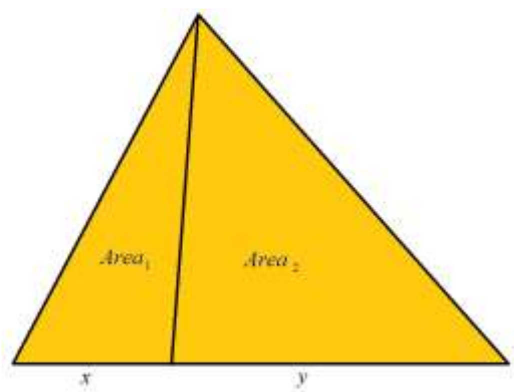


(We see $\frac{AC'}{C'B}=\frac{x}{y}$, $\frac{BA'}{A'C}=\frac{y}{z}$, and $\frac{CB'}{B'A}=\frac{z}{x}$ and the result follows.)

Comment: The equation for Menelaus' Theorem is usually written $\frac{AC'}{C'B}\times\frac{BA'}{A'C}\times\frac{CB'}{B'A}=-1$. This is to reflect the observation that if we move counter-clockwise around the triangle one of the line segments listed is traversed in the opposite sense.

A line drawn from a vertex of a triangle to a point on the opposite side of the triangle is called a *Cevian*. The name is to honor 18th-century Italian geometry Giovanni Ceva who proved a significant result about triangle Cevians. (Ceva's Theorem appears later in this essay.)

A Cevian divides a triangle into two triangles. The areas of these two triangles, which have the same height, come in the ratio of their base lengths.



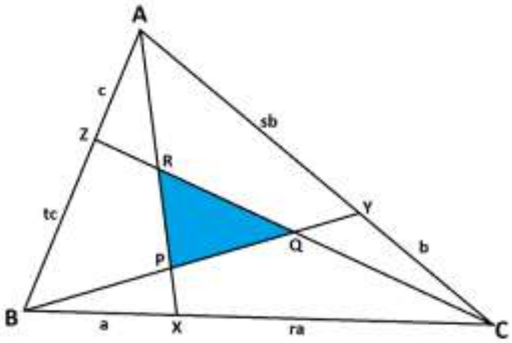
$$\frac{Area_1}{Area_2}=\frac{\frac{1}{2}x\times height}{\frac{1}{2}y\times height}=\frac{x}{y}.$$

Question: Is this result still true if the Cevian of a triangle does not lie inside the triangle? (That is, if the Cevian from a vertex meets a point on an extension of the opposite side?)

ROUTH'S THEOREM

In 1896, English mathematician Edward John Routh published a general formula for the area of the inner triangle formed by three general (inner) Cevians of a triangle.

Consider the picture labeled as shown with Cevians dividing their opposite sides in ratios r, s, and t.



If we assume $Area(ABC)$ is one square unit, then Routh proved that the area of the inner shaded triangle is:

$$Area(PQR) = \frac{(rst-1)^2}{(1+r+rt)(1+s+sr)(1+t+ts)}$$

We can prove this too!

Look at ΔABX. Observe

$$\frac{Area(ABX)}{Area(ABC)} = \frac{Area(ABX)}{1}$$
$$= \frac{a}{(1+r)a} = \frac{1}{1+r}.$$

So $Area(ABX) = \frac{1}{1+r}$.

Thus $Area(ACX) = 1 - \frac{1}{1+r} = \frac{r}{r+1}$.

Now look at ΔABX and the line CRZ. By Menelaus's Theorem:

$$\frac{AZ}{ZB} \cdot \frac{BC}{CX} \cdot \frac{XR}{RA} = 1.$$

That is,

$$\frac{1}{t} \cdot \frac{1+r}{r} \cdot \frac{XR}{RA} = 1,$$

so $\frac{XR}{RA} = \frac{rt}{1+r}$. Consequently,

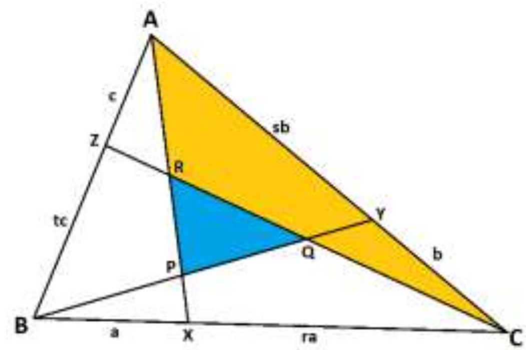
$$\frac{RA}{XA} = \frac{RA}{XR+RA} = \frac{1}{\frac{XR}{RA}+1}$$
$$= \frac{1}{\frac{rt}{1+r}+1} = \frac{1+r}{1+r+rt}.$$

This means:

$$\frac{Area(ARC)}{Area(XAC)} = \frac{1+r}{1+r+rt}.$$

We already have a formula for $Area(XAC)$, so

$$Area(ARC) = \frac{1+r}{1+r+rt} \times \frac{r}{1+r}$$
$$= \frac{r}{1+r+rt}.$$



Similarly

$$Area(BPA) = \frac{s}{1+s+sr}$$

and

$$Area(CQB) = \frac{t}{1+t+ts}.$$

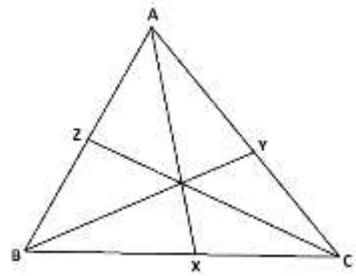
This means the area we seek is:

$$Area(PQR) = 1 - \frac{r}{1+r+rt} - \frac{s}{1+s+sr} - \frac{t}{1+t+ts}$$

which is algebraically equivalent to Routh's formula. (Feel free to double check!)

FEYNMAN'S PROBLEM: Here we have $r = s = t = 2$. Substituting into Routh's formula we get $Area(PQR) = \frac{1}{7}$.

CEVA'S THEOREM:
Following the notation of this diagram:

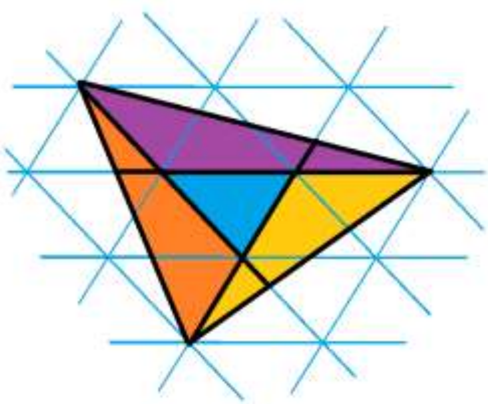


Three Cevians of a triangle are concurrent (pass through a common point) if, and only if, $\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = 1.$

Proof: The area of the inner triangle formed by three Cevians is zero if, and only if, this relation is true. (We need $rst = 1$ in Routh's formula).

SOLVING FEYNMAN'S PROBLEM WITHOUT THE BIG GUNS

There was no need for us to invoke Routh's formula to solve Feynman's problem. A simple picture does the trick!

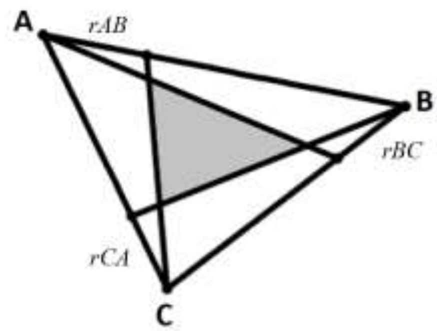


Each non-blue triangle is half a parallelogram composed of four triangles, and so has area equivalent to two blue triangles. Thus the total area of the triangle is equivalent to 7 blue triangles. Done!

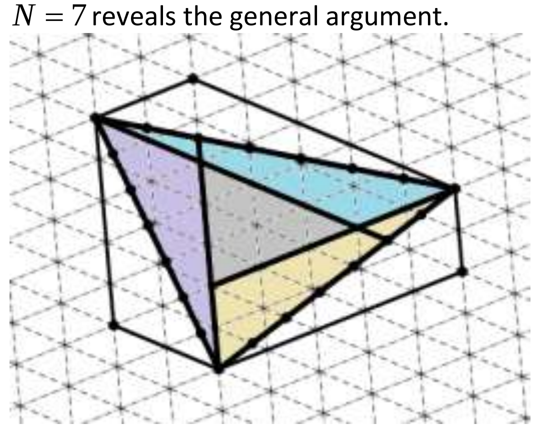
This picture is the basis of a wordless proof of Routh's formula for the case of Cevians intercepting their oppposite sides in three identical rational ratios.

SPECIAL VARIANT OF ROUTH'S THEOREM:
Let $0 < r < 1$ be a rational number. Then

$$\frac{\text{area of shaded triangle}}{\text{area of } \Delta ABC} = \frac{1-4r(1-r)}{1-r(1-r)}$$



Proof: This example with $r = \frac{k}{N}$ for $k = 2$, $N = 7$ reveals the general argument.



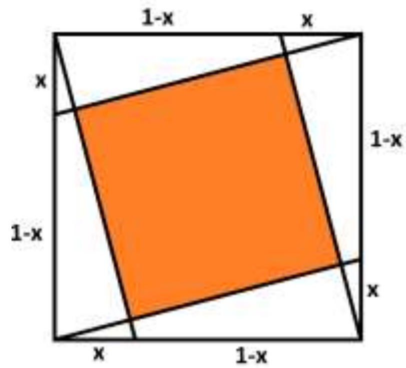
$$\frac{\text{area of inner shaded triangle}}{\text{area of } \Delta ABC}$$

$$= \frac{(N-2k)^2}{(N-2k)^2 + 3 \times \frac{1}{2} \cdot 2k(N-k)}.$$

$$= \frac{(1-2r)^2}{(1-2r)^2 + 3r(1-r)}.$$

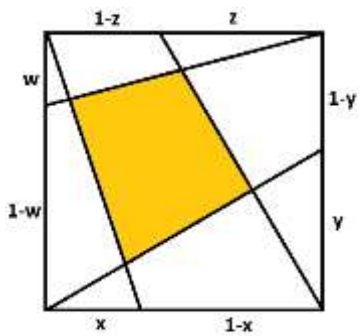
Question: How does the formula presented here line up with Routh's original formula?

CHALLENGE: Find a formula for the area of the shaded square shown drawn within a square.



RESEARCH CORNER

Given a square, is there are general formula for the area of the shaded quadrilateral shown?



Is there perhaps a formula at least for the case $x = z$, $y = w$?

Are there any area results to be had for lines drawn in rhombi? Parallelograms? General quadrilaterals?

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