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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

December 2018

THIS MONTH'S PUZZLER:

There are certainly two non-zero powers of two that sum to a power of two.

$$3^2 + 4^2 = 5^2$$
$$20^2 + 21^2 = 29^2$$
$$119^2 + 120^2 = 169^2$$

And there are three non-zero powers of three that sum to a power of three.

$$3^3 + 4^3 + 5^3 = 6^3$$
$$9^3 + 10^3 + (-12)^3 = 1^3$$
$$(-1)^3 + 73^3 + 144^3 = 150^3$$

Are there four non-zero powers of four that sum to a power of four? (Quickly: Can you see that $3^4 + 4^4 + 5^4 + 6^4 = 7^4$ can't be true?)

ON SUMS OF POWERS

In 1637, Pierre de Fermat conjectured, and alluded to possibly proving, that the sum of two positive k th powers will never equal another kth power if k is an integer greater than two. (And this was indeed proved to be the case in 1994 by Andrew Wiles.)

$$a^k + b^k \neq c^k \text{ for } k > 2.$$

In particular, there are no two positive cubes that sum to another cube. But as the opening puzzler shows, a cube can equal a sum of three other positive cubes.

In 1769 Leonhard Euler conjectured that no fourth power is a sum of three fourth powers.

$$a^4 + b^4 + c^4 \neq d^4$$

(and also that no positive fifth power is a sum of four fifth powers, no positive sixth power is a sum of five sixth powers, and so on).This was proved wrong over 200 year later. In 1986 Noam Elkies discovered that

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

along with a method for generating infinitely many more counter examples.

Two years later, Roger Frye found the only counter-example with each number presented smaller than a million.

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

There are fourth powers that are the sum of four fourth powers. A small example was discovered in 1911 by R. Norrie.

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4$$

This second example

$$955^4 + 1700^4 + 2364^4 + 5400^4 = 5491^4$$

is particularly interesting if you regard 2364^4 as $(-2364)^4$. Then this example represents a solution to the equation

$$a^4 + b^4 + c^4 + d^4 = (a+b+c+d)^4.$$

In 2008, Lee W. Jacobi and Daniel J. Madden then used this one example to generate infinitely many solutions to this more specialized equation. Thus there are infinitely many examples of (four) integers whose sum of fourth powers matches the fourth power of their sum.

MORE SUMS OF FOUTH POWERS

The method of last month's essay can be applied to this context too.

Suppose we have two sets of integers $\{a, b, c, \ldots\}$ and $\{p, q, r, \ldots\}$ satisfying this property.

$$a^4 + b^4 + c^4 + \cdots = (a+b+c+\cdots)^4$$
$$p^4 + q^4 + r^4 + \cdots = (p+q+r+\cdots)^4$$

Now look at the entries in the multiplication table created by them.

X	a	b	c	...
p	pa	pb	pc	...
q	qa	qb	qc	...
r	ra	rb	rc	...
⋮	⋮	⋮	⋮	

The sum of all these entries in the table appears when we expand

$$(a+b+c+\cdots)(p+q+r+\cdots).$$

Thus the sum of entries in this table raised to the fourth powers is

$$(a+b+c+\cdots)^4(p+q+r+\cdots)^4$$
$$=(a^4+b^4+c^4+\cdots)(p^4+q^4+r^4+)$$
$$=a^4p^4+a^4q^4+a^4r^4+b^4p^4+\cdots$$

which is the sum of each individual entry of the table raised to the fourth power.

Thus from

$$955^4+1700^4+(-2364)^4+5400^4$$
$$=(955+1700-2364+5400)^4$$

and from a second example

$$7590^4+27385^4+(-31764)^4+48150^4$$
$$=(7590+27385-31764+48150)^4$$

we obtain

$$7248450^4+26152675^4+(-30334620)^4$$
$$+45983250^4+12903000^4+46554500^4$$
$$+(-53998800)^4+81855000^4$$
$$+(-17942760)^4+(-64738140)^4$$
$$+75090096^4+(-113826600)^4$$
$$+40986000^4+147879000^4$$
$$+(-171525600)^4+26001000^4$$

$$=\begin{pmatrix}7248450+26152675-30334620\\ +45983250+12903000+46554500\\ -53998800+81855000-17942760\\ -64738140+75090096-113826600\\ +40986000+147879000\\ -171525600+260010000\end{pmatrix}^4$$

RESEARCH CORNER

Fix positive integers j, k, m, n and call a finite set of integers $\{a,b,c,...\}$ "(j,k,m,n) special" if they satisfy the following relation.

$$(a^j+b^j+c^j+\cdots)^k=(a^m+b^m+c^m+\cdots)^n$$

For example, $\{1,2,3\}$ is $(3,1,1,2)$-special since $1^3+2^3+3^3=(1+2+3)^2$, and $\{955,1700,-2364,5400\}$ is $(4,1,1,4)$-special.

Prove that if $\{a,b,c.,...\}$ and $\{p,q,r,...\}$ are each (j,k,m,n)-special, then so is their product set $\{ap,bp,cp,...,aq,bq,cq,...,ar,br,cr,...\}$.

Find examples of (j,k,m,n)-special sets for different integers j, k, m, and n.

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