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WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

December 2016

THIS MONTH'S PUZZLER

Consider the sequence of finite decimals

$$.1 = .1$$
$$.1|4 = .14$$
$$.1|4|9 = .149$$
$$.1|4|9|16 = .1506$$
$$.1|4|9|16|25 = .15085$$
$$\vdots$$

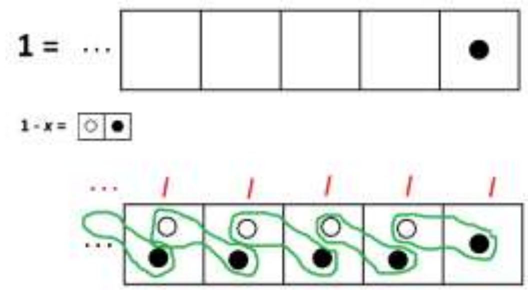
each obtained by inserting the first N square numbers in the first N decimal places and performing carries. Might the limit of this sequence be an infinite decimal with a periodic pattern?

THE GEOMETRIC SERIES FORMULA

Lovers of Exploding Dots (see www.gdaymath.com/courses) are usually smitten by the swift appearance of the geometric series formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Simply take the polynomial 1 and divide it by the polynomial $1-x$ and an infinite sum of the powers of x naturally appear.



One can alternatively conduct this division purely algebraically by repeatedly rewriting the numerator of $\frac{1}{1-x}$ so that multiples of the denominator appear.

$$\begin{aligned}\frac{1}{1-x} &= \frac{1-x+x}{1-x} = 1+\frac{x}{1-x}\\ &= 1+\frac{x(1-x)+x^2}{1-x} = 1+x+\frac{x^2}{1-x}\\ &= 1+x+\frac{x^2(1-x)+x^3}{1-x} = 1+x+x^2+\frac{x^3}{1-x}\\ &= \cdots\\ &= 1+x+x^2+x^3+x^4+\cdots\end{aligned}$$

But now one is starting to feel a tad suspicious of the $\cdots$s. What do they really mean? Are they even meaningful?

Another approach to this formula is to make an explicit leap of faith. Let's assume that the infinite sum $1+x+x^2+x^3+\cdots$ is meaningful and has an answer. Call that answer S. Then

$$\begin{aligned}S &= 1+x+x^2+x^3+\cdots\\ &= 1+x\left(1+x+x^2+\cdots\right)\\ &= 1+xS\end{aligned}$$

from which we get $S=\frac{1}{1-x}$.

This is perhaps the most honest approach. It proves: *IF the infinite sum* $1+x+x^2+\cdots$ *has an answer, then that answer must be* $\frac{1}{1-x}$. It makes no assertion as to whether or not the infinite sum is meaningful in the first place.

We reached this same conclusion in the September 2012 Curriculum Math Essay (www.jamestanton.com/?p=1072). But there we went a step further and showed that the infinite sum actually does have a definite value if x is a fraction of the form $\frac{1}{N}$ for a positive whole number N. And lo and behold, the value of that sum is $\frac{1}{1-\frac{1}{N}}=\frac{N}{N-1}$. (We practiced a paper-tearing exercise.)

In a calculus class, one goes further still proves that the sum has a meaningful answer if x is any real number between -1 and 1. (And again, that answer is $\frac{1}{1-x}$, as it must be.) The calculus argument essentially goes as follows.

> The algebra on the left shows
>
> $$1+x=\frac{1}{1-x}-\frac{x^2}{1-x}=\frac{1-x^2}{1-x}$$
>
> $$1+x+x^2=\frac{1}{1-x}-\frac{x^3}{1-x}=\frac{1-x^3}{1-x}$$
>
> and in general
>
> $$1+x+x^2+\cdots+x^{n-1}=\frac{1-x^n}{1-x}.$$
>
> Now, by an infinite sum $1+x+x^2+\cdots$ we mean a limit value of the finite sums $1+x+x^2+\cdots+x^n$ as n grows, if a limit value exists. So we are asking if $\frac{1-x^n}{1-x}$ has a limit value as n grows.
>
> This depends on whether or not x^n has a limit value as n grows. And it does if $-1<x<1$: larger and larger powers of a small number approach zero.
>
> Moreover, we have

$$1+x+x^2+\cdots = \text{limit value of } \frac{1-x^n}{1-x}$$
$$= \frac{1-0}{1-x}$$
$$= \frac{1}{1-x}$$

if $-1 < x < 1$.

However, there are certainly some values of x for which the geometric series formula is decidedly meaningless! Put in $x=2$ for example, and we are supposedly led to conclude that

$$1+2+4+8+16+\cdots$$

adds to $\frac{1}{1-2} = -1$. This is absurd.

Put in $x=10$ and we get $1+10+100+1000+\cdots = -\frac{1}{9}$.

That seems even more absurd to me.

FORMAL ALGEBRA

But Exploding Dots and the algebra argument both show, as a statement of pure algebra, without regard to meaning in arithmetic, $\frac{1}{1-x}$ really does "want to be" $1+x+x^2+x^3+\cdots$. That $1+x+x^2+\cdots$ really does behave algebraically the same way $\frac{1}{1-x}$ does.

What does it mean for $1+10+100+1000+\cdots$, say, to behave the same way algebraically as $-\frac{1}{9}$? Can I multiply $1+10+100+1000+\cdots$ by 9 and get the answer -1?

Maybe we can argue we can!

$$9+90+900+9000+\cdots$$
$$= 10-1+100-10+1000-100+10000-1000+\cdots$$
$$= -1$$

Of course, one might argue that nonsense begets nonsense, that one can entertain all sorts of delightful folly if one starts with a meaningless premise.

But there is some value to this meaningless work. For example, Exploding Dots (or pure algebra) shows that $\frac{1}{1-x-x^2}$ equals $1+x+2x^2+3x^3+5x^4+8x^5+13x^6+\cdots$. All the algebraic properties of the Fibonacci numbers are encoded in the formal algebraic structure of $\frac{1}{1-x-x^2}$. So if one is willing to explore the formal algebraic manipulations of infinite sums, without regard to the arithmetical meaning, one can discover all sorts of arithmetic structures among the Fibonacci numbers.

OTHER MEANINGS

But, of course, we are still perturbed by a statement of the form $1+2+4+8+\cdots = -1$, for example. Can such a statement have arithmetic meaning?

Clearly not in the ordinary way we think of arithmetic. Is there an extraordinary way to view matters?

We tend to view numbers as spaced apart on the number line additively. Walk one step to the right of 0 and we end up at position 1. Now add to that two steps and we end up position 3. Now add four steps, position 7. And so on. The sum $1+2+4+8+\cdots$, in this viewpoint, takes infinitely far to the right of 0 on the number line. It cannot land us at position -1 just one pace to the left of zero.

But let's think of numbers multiplicatively. In particular, since we are focusing on the sum $1+2+4+8+\cdots$ let's think of factors and multiples of powers of two.

Now 0 is a highly divisible number. It is the most divisible number of all. With regard to just two-ness it can be divided by 2 once, in fact twice, in fact thrice, in fact, you can divide 0 by two as many times as you like – and still keep going.

With regard to two-ness, the number 8 is somewhat zero-like: you can divide it by two three times. But 32 is even more zero-like: you can divide it by two five times. And 2^{100} is even more zero-like still.

So in this sense, 2^{100} is a number very close to 0. It is closer than 32 is, which is closer than 8. The bigger power of two we have, the closer to 0 it is. The number 1 is not very close to zero at all: it cannot be evenly divided by two even once.

So, in this context, could $1+2+4+8+\cdots$ possibly be -1?

Well

$$
\begin{aligned}
1+2&=3 &&=4-1\\
1+2+4&=7 &&=8-1\\
1+2+4+8&=15 &&=16-1\\
&\vdots\\
1+2+\cdots+2^{99}& &&=2^{100}-1
\end{aligned}
$$

These finite sums grow to become "a number very close to zero, minus one." In the limit, the infinite sum thus has value $0-1=-1$, just as our formal arguments said it would be. (If $1+x+x^2+\cdots$ has an answer, then that answer has to be $\frac{1}{1-x}$.)

> **Challenge:** If $1+2+4+8+\cdots$ equals -1, then $(1+2+4+8+\cdots)^2$ should equal 1. Show that it does in this multiplicative thinking of "closeness."
>
> [Start by showing that sum squared equals $1\cdot 1+2\cdot 2+3\cdot 4+4\cdot 8+5\cdot 16+\cdots$.]

RESEARCH CORNER

Mathematicians call this two-ness reimagining of the number line 2*-adic arithmetic.* It seems natural to try to put numerical values to the distances of numbers from 0.

Recall here we want 2^{100} to be much closer to 0 than 32. One way to quantify these distances is to declare

$$d\left(0,2^{100}\right)=\frac{1}{2^{100}}$$

$$d(0,32)=\frac{1}{2^5}$$

and in general

$$d(0,N)=\frac{1}{2^m}$$

if N can be evenly divide by two m times, but not $m+1$ times. So $d(0,12)$, for instance, is $\frac{1}{2^2}$.

Since 15 and 27 differ by 12, it seems natural to declare the distance between these two numbers to be the same as the distance between 0 and 12.

$$d(15,27)=d(0,12)=\frac{1}{2^2}.$$

In the same way $d(-3,9)$ should also equal $d(0,12)$. In general,

$$d(x,y)=d(0,|x-y|).$$

We'll also set $d(x,x)=0$ and declare $d(x,y)=d(y,x)$ for all x and y.

> **Challenge:** Prove that $d(x,y)+d(y,z)\ge d(x,z)$.

[Mathematicians strongly feel that a "good" notion of distance should satisfy the triangle inequality.]

We have actually declared the following.

If $N = 2^m k$ with k odd, then $d(0, N) = \frac{1}{2^m}$.

This suggests how to we could extend our new notion of distance to fractions too.

If $r = 2^m \frac{a}{b}$, with a and b odd integers, and m possibly a negative integer, then declare $d(0, r) = 2^{-m}$.

We have, for example,

$$d\left(\frac{5}{6}, \frac{1}{3}\right) = d\left(0, \frac{1}{2}\right) = 2.$$

Challenge: Prove that the triangle inequality still holds.

Now for the research challenge.

How would you extend this notion of distance to all real numbers? Can you? What is a real number?

[Mathematicians have thought about this issue too. This could be an internet research project. Look up p -*adic numbers*.]

Challenge: Does $1 + 10 + 100 + 1000 + \cdots$ really converge to $-\frac{1}{9}$ in a 10-adic arithmetic system?

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