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WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

DECEMBER 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep, joyous, and real mathematical doing I'll happily mention it here.*

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Look for WITHOUT WORDS and MORE WITHOUT WORDS – with classroom posters! – soon to be released in the U.S.

OPENING PUZZLE: The A series of international paper sizes sets A4 paper as 297mm by 210 mm (which is approximately 11 ¾ inches by 8 ¼ inches). Why the numbers 297 and 210?

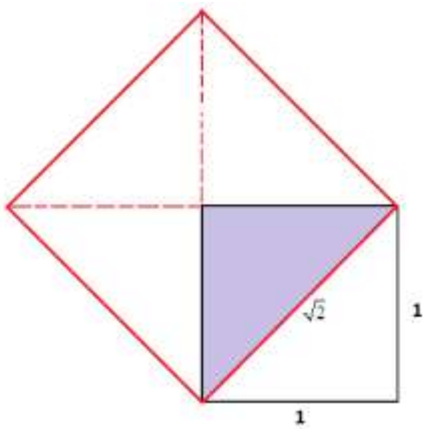
Comment: The US does not follow international standards: A4 paper here is 11 inches by 8 ½ inches. Why doesn't the US follow international standards?

THE SQUARE ROOT OF TWO

The square root of two is defined to be the side length of a square of area two. Do we know that such a square exists?

We can certainly draw a square of side length one unit (use your favorite unit of length: an inch, a meter, a furlong) and hence draw a square of area one square unit. But how do you draw a square of area two square units? This is not a trivial question and takes an epiphany to see how to do it.

The answer is to draw a tilted square on the diagonal of the unit square.



This tilted square is composed of four right triangles, each of area half a square unit, and so has total area 2. The square root of two does indeed exist and it matches the length of the diagonal of a unit square.

CHALLENGE: Draw a picture to make it clear that the square root of three actually exists.

We all "know" that the square root of two is an irrational number, that $\sqrt{2}$ cannot be written as a fraction.

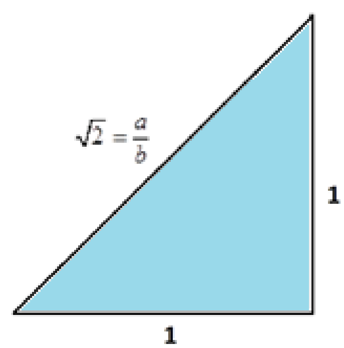
But how do we know? (And why do so many school curricula insist on testing students on "knowing" that $\sqrt{2}$ is an irrational number when absolutely no proof of this assertion is ever proffered?)

One approach to proving the irrationality of $\sqrt{2}$ is to assume that the number <u>is</u> a fraction and then see if something goes wrong mathematically in believing this the case. If indeed the mathematics does go awry, we can only conclude then that our beginning assumption was wrong.

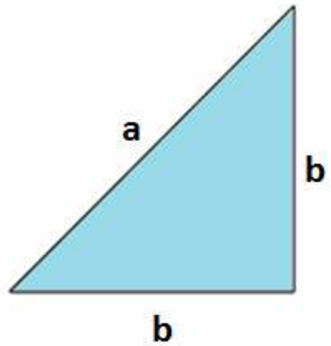
So let's believe that the square root of two is rational for a moment and see what happens. Let's assume we can write $\sqrt{2} = \frac{a}{b}$ for a pair of integers a and b.

Actually, we may as well assume that the particular integers we use, a and b, are as small as possible. By this, I mean, we can assume that we've canceled out any common factors in the numerator and denominator and so we are writing $\sqrt{2} = \frac{a}{b}$ as a reduced fraction.

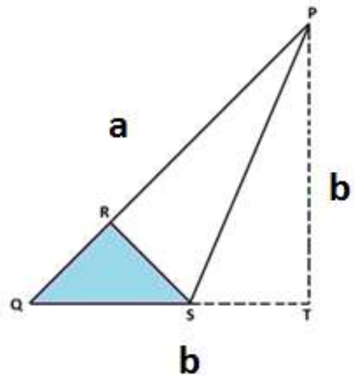
So with this belief in place, we can say we have a right isosceles triangle with side lengths 1 and hypotenuse $\frac{a}{b}$.



Let's scale this picture up by a factor b. This then gives a right isosceles triangle with side lengths b and hypotenuse a, all integers. And, moreover, these are the smallest integers possible for the sides of a right isosceles triangle.



Great. Now fold the right isosceles triangle.



In this labeled picture, edge PT is folded over to PR. Length RS equals length ST.

Angle T was a right angle, so the angle at R is a right angle as well. Angle Q is 45°. So the shaded triangle, ΔQRS, is another right isosceles triangle. It can't have integer side lengths as the numbers a and b were already the smallest integer sides of such a triangle.

But look. The sides of ΔQRS are integers! We have

$$\begin{aligned} QR &= QP - RP = a - b \\ RS &= QR = a - b \\ QS &= b - ST \\ &= b - RS = b - (a-b) = 2b - a \end{aligned}$$

The math has gone awry. We have a right triangle whose sides both must be and can't be all integers.

All our reasoning was absolutely solid and correct. The only thing that can be wrong with our work was believing at the very beginning that $\sqrt{2}$ can be written as a fraction. It must be the case then that $\sqrt{2}$ is not a fraction.

CHALLENGE: Reverse this process. Start with a right triangle QRS with side lengths d, d, and c, and pretend it arose from this folding exercise. Show that the larger right triangle whence it came has dimensions $c+d$, $c+d$, $c+2d$.

CHALLENGE: Motivated by the previous challenge, show that if a fraction $\frac{c}{d}$ is an approximation for $\sqrt{2}$, then $\frac{c+2d}{c+d}$ is a better one.

For example, starting with the (lousy) approximation $\frac{1}{1}$ for $\sqrt{2}$, this process gives the sequence of fractions:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \ldots$$

One can check on a calculator that the squares of these fractions do indeed seem to get closer and closer to the value 2.

A4 PAPER

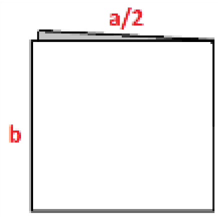
The A series of international paper sizes follows two criteria: the paper size must be pleasant to handle and pleasant to look at, and when the paper is folded in half, one should obtain a smaller rectangle of paper of the same proportions as the original.

What size rectangular paper meets these criteria?

The second criterion is a mathematical one, so let's look at that one first. We want a rectangle of some proportions so that half that rectangle has the same proportions.

Suppose we start with a rectangle a units on the long side and b units on the short side. Then its long-to-short proportions are $\frac{a}{b}$.

If we fold the rectangle in half, we get a rectangle with long side b and short side $a/2$. It has long-to-short proportions $\frac{b}{a/2}$.



The international standards want these to be the same proportion. That is, they want

$$\frac{a}{b} = \frac{b}{a/2}.$$

This can be rearranged to read:

$$\left(\frac{a}{b}\right)^2 = 2$$

or equivalently

$$\frac{a}{b} = \sqrt{2}.$$

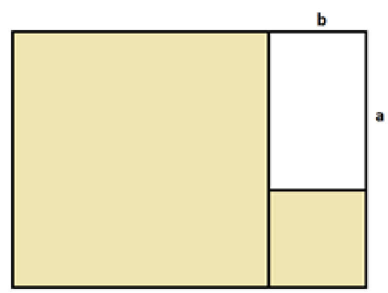
So we want one side of the paper to be longer than the other by a factor of $\sqrt{2}$. This meets the second criterion. But this is problematic: the numbers a and b should be a whole number of millimeters, or at least as easy fraction of millimeters, so that manufacturers have the means to measure and make the paper. But we've proved that $\sqrt{2}$ is not a fraction and so no such numbers a and b exist!

Nonetheless, the folk if international standards went with the values 297 mm and 210 mm for a and b. Why these numbers?

These dimensions do give a rectangular sheet that meets the first criterion – being pleasing to handle. But what about that that second criterion? $\frac{297}{210} = \frac{99 \times 3}{70 \times 3} = \frac{99}{70}$ is not the square root of two.

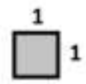
THEON OF SMYRNA

Scholars since the time of antiquity have been thinking about and playing with the number the square root of two. Theon of Smyrna (ca 140 CE), for example, was fully aware that if a rectangle of sides a units and b units, has proportions that approximate $\sqrt{2}$, then drawing a square on the small side of that rectangle followed by another square on the long side of what results gives a new, larger rectangle with proportions that more closely approximate $\sqrt{2}$.



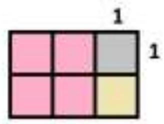
(That is, if $\frac{a}{b}$ approximates $\sqrt{2}$, then $\frac{a+2b}{a+b}$ is a better approximation. Sound familiar?)

For example, if we start with a 1×1 square, representing the (lousy) approximation $\frac{1}{1} = 1.0000....$ to the square root of 2,



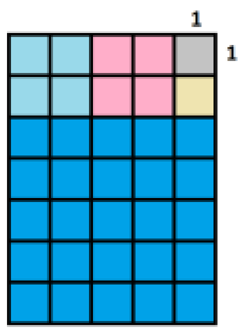
$$\frac{1}{1} = 1.000000....$$

then Theon's method suggests that a 3×2 rectangle gives a better approximation:



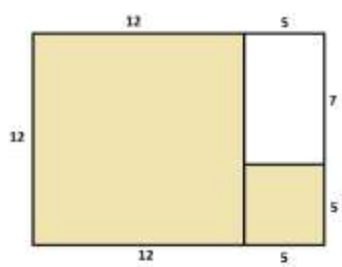
$$\frac{3}{2} = 1.500000....$$

and a 7×5 rectangle a better approximation still



$$\frac{7}{5} = 1.400000...$$

Next comes the a 17×12 rectangle



$$\frac{17}{12} = 1.416666...$$

We get the sequence of fractions:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \ldots$$

which is today known as Theon's Ladder.

We can prove that this sequence converges to $\sqrt{2}$.

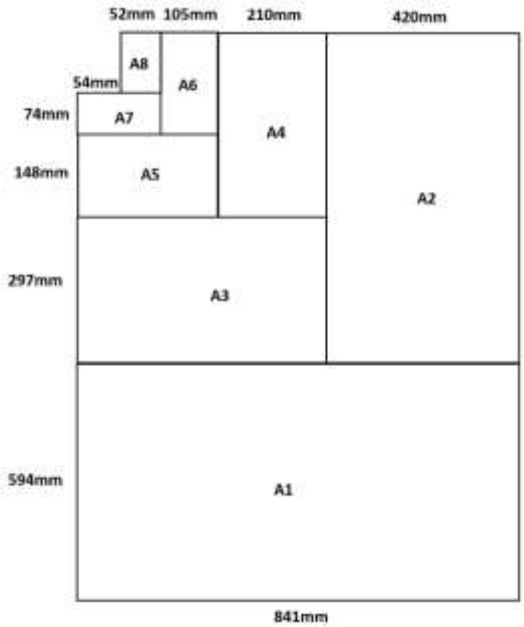
Proof: Suppose $\frac{a}{b}$ is a fraction that approximates the square root of two. Then $\left(\frac{a}{b}\right)^2 = 2 + \varepsilon$, for some error ε. The next term in Theon's Ladder is $\frac{a+2b}{a+b}$, and

$$\left|2 - \left(\frac{a+2b}{a+b}\right)^2\right| = \left|2 - \frac{a^2+4ab+4b^2}{a^2+2ab+b^2}\right|$$
$$= \left|\frac{a^2-2b^2}{a^2+2ab+b^2}\right|$$
$$= \left|\frac{\varepsilon}{(a+b)^2}\right| < \frac{|\varepsilon|}{4}$$

since, in Theon's Ladder, a and b are always each at least 1. Thus $\left(\frac{a+2b}{a+b}\right)^2$ differs from 2 by less than quarter the previous error. □

The folk who set international A4 paper size knew about these fractions and used the approximation $\frac{99}{70}$ to create a piece of paper with proportions that approximate the square root of 2. They tripled 99 and 70 and used the dimensions 297 mm and 210 mm. This is the size that seemed pleasing to handle.

All the remaining standard A paper sizes, A1, A2, A3, and so on, are close approximations to scaled versions of this A4 size .They (almost) fit together to make their own tiling pattern.



By the way there are international B and C series paper sizes too. They are for posters and for envelopes.

AN INFINITE FRACTION FOR $\sqrt{2}$

Each fraction in Theon's Ladder can be rewritten in a beautiful way:

$$\frac{1}{1}=\frac{1}{1}$$

$$\frac{3}{2}=1+\frac{1}{2}$$

$$\frac{7}{5}=1+\cfrac{1}{2+\cfrac{1}{2}}$$

$$\frac{17}{12}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}$$

$$\frac{41}{29}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}$$

And so on.

CHALLENGE: Show that $\frac{a+2b}{a+b}$ equals $1+\cfrac{1}{1+\cfrac{a}{b}}$. Use this to explain why the fractions in Theon's Ladder follow the indicated pattern.

Since the fractions in Theon's Ladder converge to $\sqrt{2}$, this suggests that the "ultimate term" of this pattern, the infinite fraction $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}}$, equals $\sqrt{2}$.

CHALLENGE:

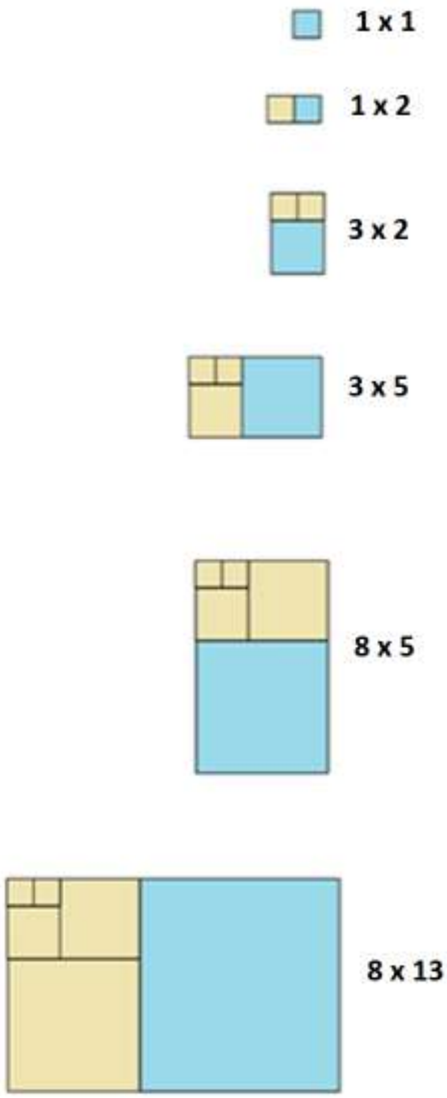
Show that $\sqrt{2}=1+\frac{1}{1+\sqrt{2}}$.

Now substitute this formula into itself to get $\sqrt{2}=1+\cfrac{1}{1+\left(1+\cfrac{1}{1+\sqrt{2}}\right)}$.

Substitute the original formula into this expression, and keep doing this. What do you seem to conclude?

RESEARCH CORNER

Stacking squares together in different patterns to create larger and larger rectangles gives approximations to different irrational numbers. For example, start with the unit square and add just one square at each stage of the game this time, one square to the longer side of the rectangle one has.



This construction pattern gives us rectangles with sides the Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots.$. The ratios of these side lengths, $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$, approximate the Golden Ratio $\frac{1+\sqrt{5}}{2} \approx 1.618$. (Can you prove all these claims?)

Explore other square tiling patterns and the sequences that result to approximate irrational numbers.

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