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WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

DECEMBER 2014

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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The MAA's Curriculum Inspirations project is still mighty strong! Two new puzzles and videos each and every week. (Wow!) Check out: www.maa.org/ci.

This month let's explore two tricky trigonometry questions. There's plenty here for a full essay of challenging thinking!

PUZZLE 1:

Is $\sin\left(1^{\circ}\right)$ rational or irrational?

Is $\cos\left(1^{\circ}\right)$ rational or irrational?

Is $\tan\left(1^{\circ}\right)$ rational or irrational?

PUZZLE 2:

Is $\sin\left(x \text{ degrees } + x \text{ radian }\right)$ periodic?

Is $\sin\left(x \text{ degrees}\right)+\sin\left(x \text{ radian}\right)$ periodic?

Comment: The first function given is ambiguous: Is the input number, overall, a number of degrees or a number of radians? For example, for $x=1$, we have:

$$1 \text{ degree} + 1 \text{radian}$$
$$\approx 1 + 57.3 = 58.3 \text{ degrees}$$

and

$$1 \text{ degree} + 1 \text{radian}$$
$$\approx \frac{\pi}{180} + 1 \text{ radian.}$$

Answer the question both ways, first regarding the function with inputs in degrees and then as a function with inputs in radian.

BASIC TRIGONOMETRIC VALUES

We certainly know the trigonometric function values at five basic angles: 0°, 30°, 45°, 60°, and 90°. (See the video http://www.jamestanton.com/?p=875 for a very handy mnemonic for these!)

And the double angle formulas:

$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$

allow us to compute the exact trigonometric values for more angles.

For example, from $1 - 2\sin^2 x = \cos(2x)$ we get:

$$\sin(15^\circ) = \sqrt{\frac{1-\cos(30^\circ)}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$\cos(15^\circ) = \sqrt{1-\sin^2(15^\circ)} = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

and $\tan(15^\circ) = \frac{\sqrt{3}-1}{\sqrt{3}+1}$.

Carrying on we can compute exact trigonometric values for $7\frac{1}{2}^\circ$, $3\frac{3}{4}^\circ$, $1\frac{7}{8}^\circ$, and $\frac{15}{16}^\circ$ to see that each of these values is irrational. But alas, this approach won't tell us about the trigonometric values of 1°.

IS $\sin(1^\circ)$ RATIONAL OR IRRATIONAL?

We saw in this month's Curriculum Math essay that the double, triple, n th tuple angle formulas for sine and cosine can be summarized in the single statement:

$$\cos(Nx) + i\sin(Nx) = (\cos x + i \sin x)^N.$$

Expanding gives:

$$\cos(Nx) + i\sin(Nx)$$
$$= \cos^N x - \binom{N}{2}\cos^{N-2} x \sin^2 x + \binom{N}{4}\cos^{N-4} x \sin^4 x - \cdots$$
$$+ i(\text{stuff})$$

Using $\sin^2 x = 1 - \cos^2 x$ this shows:

$\cos(Nx)$ can be expressed as a combination of powers of $\cos x$ using integer coefficients.

So, in particular, $\cos(N^\circ)$ can always be written as a combination of powers of $\cos(1^\circ)$ using integer coefficients. So if $\cos(1^\circ)$ is rational, so too is $\cos(N^\circ)$ for every value of N. But $\cos(30^\circ) = \frac{\sqrt{3}}{2}$ is irrational. This means that $\cos(1^\circ)$ must, instead, be irrational.

If $\sin(1^\circ)$ is rational, then so too is $\sin^2(1^\circ)$ and $\cos(2^\circ) = 1 - 2\sin^2(1^\circ)$. Since $\cos(2N^\circ)$ can be written as combination of powers of $\cos(2^\circ)$ with integer coefficients, this means that

$\cos\left(2N^\circ\right)$ is rational for all N. But it is not! Choose $N=15$. It must be then that $\sin\left(1^\circ\right)$ is irrational.

From $\tan\left(\alpha+\beta\right)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ it follows that:

$$\tan\left(30^\circ\right)=\frac{\tan 1^\circ+\tan 29^\circ}{1-\tan 1^\circ \tan 29^\circ}$$

with

$$\tan\left(29^\circ\right)=\frac{\tan 1^\circ+\tan 28^\circ}{1-\tan 1^\circ \tan 28^\circ}$$

with

$$\tan\left(28^\circ\right)=\frac{\tan 1^\circ+\tan 27^\circ}{1-\tan 1^\circ \tan 27^\circ}$$

and so on.

This shows that we can write $\tan\left(30^\circ\right)$ as a "tower of fractions" based on $\tan\left(1^\circ\right)$. So if $\tan\left(1^\circ\right)$ is rational, so too is $\tan\left(30^\circ\right)=\dfrac{1}{\sqrt{3}}$, which it is not. Thus $\tan\left(1^\circ\right)$ is irrational.

> **Question:** Are sine, cosine, and tangent of 1 radian rational or irrational?

ON PERIODICITY

The trigonometric function $\sin\left(ax\right)$ has

period $\dfrac{360}{a}$ if the inputs are in degrees,

period $\dfrac{2\pi}{a}$ if the inputs are in radian.

If we work with degrees, then:

$$\sin\left(x\,\text{degrees} + x\ \text{radian}\right) = \sin\left(\left(1+\frac{180}{\pi}\right)x\ \text{degrees}\right)$$

is periodic with period $\dfrac{360}{1+\dfrac{180}{\pi}}$.

Working in radian:

$$\sin\left(x\,\text{degrees} + x\ \text{radian}\right) = \sin\left(\left(\frac{\pi}{180}+1\right)x\ \text{radian}\right)$$

is periodic with period $\dfrac{2\pi}{\dfrac{\pi}{180}+1}$.

For either interpretation the function is, for sure, periodic.

Now consider:

$$\sin\left(x\,\text{degrees}\right)+\sin\left(x\ \text{radian}\right).$$

This is a sum of two periodic functions, one of period 360 and the other of period 2π. Does this mean that it too is periodic? (If so, what period does it have?)

On the other hand, the two individual periods feel somewhat "incompatible": 360 and 2π form an irrational ratio. Does that ruin hopes for periodicity?

> **Exercise:** Suppose f is a function with period a and g is a function with period b. (So $f\left(x+a\right)=f\left(x\right)$ and $g\left(x+b\right)=g\left(x\right)$ for all real values x.) The values a and b need not be smallest possible periods for these functions.
>
> a) Show that if a and b are integers, then $f\left(x+ab\right)+g\left(x+ab\right)=f\left(x\right)+g\left(x\right)$ for all real values x. That is, show that $f+g$ is periodic with a period of ab.

b) Show that if, instead, just $\frac{a}{b}$ equals a ratio of integers, then $f+g$ is periodic. (Can you identify a period?)

Comment: The sine function has periods $2k\pi$ for $k=1,2,3,\ldots$ with 2π being the smallest ("fundamental") period for this function. But not all periodic functions have a smallest period! Consider Dirichlet's famous function as an example:

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irratrional}\end{cases}$$

Every positive rational number is a period of this function, and there is no smallest period.

Now let's tackle the periodicity or non-periodicity of

$$\sin(x\,\text{degrees})+\sin(x\text{ radian}).$$

WARNING: HEAVY MATH THINKING IS ABOUT TO HAPPEN. VERY HEAVY!

Let's look at the numbers 360 and 2π. Their ratio is certainly an irrational value. But we can approximate the ratio to any degree of accuracy we desire as a ratio of two integers simply by truncating its decimal expansion. For example, to get a rational approximation correct to third decimal place, write:

$$\frac{360}{2\pi}=57.295779513....$$
$$\approx 57.295=\frac{57295}{1000}.$$

Actually, a more delicate argument shows that we can always find integers m and n so that $m360$ and $n2\pi$ differ by less than any previously prescribed amount. For example, choosing $m=71$ and $n=4028$ gives two quantities that differ by less than 0.01:

$$71\times 360-4068\times 2\pi=0.00217....$$

Now suppose

$$\sin(x\,\text{degrees})+\sin(x\text{ radian})$$

is indeed periodic with some period L, a number different from zero. Then:

$$\sin((x+L)\text{ degrees})+\sin((x+L)\text{ radian})$$
$$=\sin(x\,\text{degrees})+\sin(x\text{ radian})$$

and so

$$\sin((x+L)\text{ degrees})-\sin(x\,\text{degrees})$$
$$=\sin(x\text{ radian})-\sin((x+L)\text{ radian}).$$

for all real values x.

Let f be the function given by

$$f(x)=\sin((x+L)\text{ degrees})-\sin(x\,\text{degrees})$$

or equivalently by

$$f(x)=\sin(x\text{ radian})-\sin((x+L)\text{ radian}).$$

Notice that the first defining expression for f makes it clear that f has a period of 360. The second shows that f also has a period of 2π. Thus we notice:

$$f(x+71\times 360-4068\times 2\pi)$$
$$=\sin((x+L+71\times 360-4068\times 2\pi)\text{ degrees})$$
$$-\sin((x+71\times 360-4068\times 2\pi)\text{ degrees})$$
$$=\sin((x+L-4068\times 2\pi)\text{ degrees})$$
$$-\sin((x-4068\times 2\pi)\text{ degrees})$$
$$=f(x-4048\times 2\pi)$$
$$=\sin((x-4068\times 2\pi)\text{ radian})$$
$$-\sin((x+L-4068\times 2\pi)\text{ radian})$$
$$=\sin(x\text{ radian})-\sin((x+L)\text{ radian})$$
$$=f(x).$$

So f also has a period of $71\times360-4068\times2\pi$, a period less than 0.01.

In fact, by working with different combinations $m360-n2\pi$ we can argue that f has a period smaller than <u>any</u> small number we care to name.

Now f is just a difference of two sine functions, and so its graph, unlike the Dirichlet function, is a nice smooth curve. And apparently it is one that oscillates with arbitrarily small periods! The only smooth continuous curve with arbitrarily small periods is a constant function. It must be that $f(x)=k$ for some fixed value k for all real numbers x.

Now $f(x)=k$ reads:

$$\sin((x+L)\text{ degrees})-\sin(x\text{ degrees})=k$$

for all real values x.

So

$$\sin((x+L)\text{ degrees})=\sin(x\text{ degrees})+k$$

and

$$\begin{aligned}\sin((x+2L)\text{ degrees})&=\sin((x+L)\text{degrees})+k\\&=\sin(x\text{ degrees})+2k.\end{aligned}$$

and

$$\begin{aligned}\sin((x+3L)\text{ degrees})&=\sin((x+2L)\text{degrees})+k\\&=\sin(x\text{ degrees})+3k.\end{aligned}$$

and so on, to show that

$$\sin((x+rL)\text{ degrees})=\sin(x\text{ degrees})+rk.$$

for all positive integers r.

Now if k is different from zero, then we can choose a value of r so that rk has magnitude over a million. Then

$$|\sin((x+rL)\text{ degrees})-\sin(x\text{ degrees})|>1000000$$

But this is absurd as values of the sine function always lie between -1 and 1, and so the difference of two sine values has magnitude no more than 2. It must be that $k=0$. That is, it must be that our function f has constant value zero.

So

$$\sin((x+L)\text{degrees})-\sin(x\text{degrees})=0$$

and

$$\sin(x\text{ radian})-\sin((x+L)\text{ radian})=0$$

for all real values x.

That is, we have:

$$\sin((x+L)\text{degrees})=\sin(x\text{degrees})$$

$$\sin((x+L)\text{radian})=\sin(x\text{radian})$$

for all real x.

So L is a multiple of 360 and a multiple of 2π. That is,

$$L=a360$$

$$L=b2\pi$$

for some integers a and b. We then have:

$$\frac{360}{2\pi}=\frac{b}{a},$$

showing that $\frac{360}{2\pi}$ is rational, which it is not!

So our very beginning assumption must be incorrect. That is, there is no number L which is a period for the function:

$$\sin(x\text{degrees})+\sin(x\text{ radian}).$$

That is, this function is <u>not</u> periodic.

Question: What does the graph of $\sin(x\text{degrees})+\sin(x\text{ radian})$ look like?

FOR THOSE WHO KNOW CALCULUS: There is a much easier way to show that $\sin(x\text{degrees})+\sin(x\text{ radian})$ is non-periodic. Use the fact that the first and second derivatives of a periodic function must also be periodic. (Is this obvious?) I

RESEARCH CORNER: FIX 1
Can you prove that it is indeed possible to always find integers m and n so that the difference $|m360 - n2\pi|$ is smaller than any small amount you might care to name?

RESEARCH CORNER: FIX 2
In our work we made the claim:

If a continuous function is periodic and has periods of sizes smaller than any small value you care to name, then the function must be a constant function.

Is this obvious? What property of continuity makes this claim true?

RESEARCH CORNER 3: The proof presented in this essay can be mimicked to establish:

Suppose f is a continuous function with period a and g is a continuous function with period b, and the ratio $\frac{a}{b}$ is irrational. Then the function $f+g$ cannot be periodic.

Is there an example of two non-continuous periodic functions with periods in an irrational ratio whose sum is periodic?

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