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WHAT HO! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

AUGUST 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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Free talk by yours truly. ***A Dozen Proofs that 1 = 2: An Accessible and Quirky Overview of Mathematics!"***
Ideal for educators and students.
Saturday August 8, 1 – 1:50 p.m.
Salon 2, Marriott Wardman Park, D.C.
(Opposite the zoo on Connecticut Ave.)

OPENING PUZZLE: The following Numberphile video, *The "Everything" Formula*, is making a bit of a buzz: https://www.youtube.com/watch?v=_s5RFgd59ao**.** The video is dynamic, exciting, deeply mysterious, and a tad frustrating: What does the formula actually mean and why does it work the way it does? (And what made its inventor think of it?)

$$\frac{1}{2} < \left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor \cdot 2^{-17\lfloor x \rfloor - \mathrm{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor$$

Tupper's Self-Referential Formula

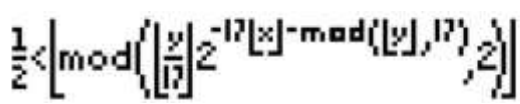
This month's puzzle: Watch the video and demystify this formula!

TUPPER'S FORMULA

In his 2001 paper on the computer science subtleties of printing graphics, *Reliable Two-Dimensional Graphing Methods for Mathematical Formulae with Two Free Variables*(http://www.dgp.toronto.edu/people/mooncake/papers/SIGGRAPH2001_Tupper.pdf), Jeff Tucker slipped in his figure 13, without comment, a curious inequality:

$$\frac{1}{2} < \left\lfloor \operatorname{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor \cdot 2^{-17\lfloor x \rfloor - \operatorname{mod}(\lfloor y \rfloor, 17)}, 2\right)\right\rfloor$$

He, and the video, claims that if one colors each pixel that contains an integer point (x, y) that makes the inequality true, then the following plot appears:



It appears in the window of the plot with $0 \le x \le 106$, $k \le y \le k+17$. Here k is the 543-digit number $960939\cdots719$.

The video then goes further in its astounding claims:

Draw ANY picture you desire of black and white pixels in a $[0,106]$*-by-*$[0,16]$ *rectangle. Then your very picture appears in the plot of this inequality AND we can tell you where to find it!*

(So your name with a happy face drawn next to it appears in the output of this plot!)

How could this possibly be?

How on Earth did Tupper come up with this remarkable formula?

UNDERSTANDING THE NOTATION

There are two pieces of notation in Tupper's inequality that don't appear in the world of high-school mathematics.

Integer rounding:

For a real number x the notation $\lfloor x \rfloor$ represents the largest integer less than or equal to x. ("Round down.") For example, $\lfloor 14.8 \rfloor = 14$, $\lfloor \pi \rfloor = 3$, $\lfloor 248 \rfloor = 248$, $\lfloor -1.1 \rfloor = -2$, and $\lfloor -10\pi \rfloor = -32$.

Just for completeness: The smallest integer greater than or equal to x is denoted $\lceil x \rceil$ ("round up") and x simply rounded to the nearest integer is denoted $\langle x \rangle$. (Show that this equals $\left\lfloor x + \frac{1}{2} \right\rfloor$.) The quantity $\{x\} = x - \lfloor x \rfloor$ is the *fractional part* of x.

The mod function:

For a real number x and positive integer b write $x = mb + r$, an integer multiple of b plus a remainder $0 \le r < b$. Then $\operatorname{mod}(x, b) = r$, this remainder. (So $\operatorname{mod}(x, b)$ is the "amount over" x is from the closest multiple of b just below it.) For example:

$$\operatorname{mod}(\pi, 2) = 1.1415926...$$
$$\operatorname{mod}(17.9, 3) = 2.9$$
$$\operatorname{mod}(973, 100) = 73.$$

If we are working with pixels positioned at integer coordinates, then we can assume x and y are integers in Tupper's inequality. (Tupper was dealing with the subtle issues of misalignment.) In this case, $\lfloor x \rfloor = x$ and $\lfloor y \rfloor = y$, and the inequality reads

$$\left\lfloor \operatorname{mod}\left(\frac{\left\lfloor \frac{y}{17} \right\rfloor}{2^{17x+(\operatorname{mod}(y,17))}}, 2 \right) \right\rfloor > \frac{1}{2}$$

This still looks mighty strange!

THE MOD FUNCTION IN COMPUTER SCIENCE

The world of computer science is based on the concept of electric switches either being on or off. Thus arithmetic in base two, 0 s and 1s, is appropriate (well, forced!) for work in this field. Just as every number in base ten can be written as a combination of powers of ten using the digits 0, 1, …, 9, every number can be written as a combination of powers of two using the digits 0 and 1.

For example, $273 = 2\times10^2 + 7\times10 + 3\times1$ and is written in base ten as "273."
For base two:

$$273 = 1\times2^8 + 0\times2^7 + 0\times2^6 + 0\times2^5 + 1\times2^4 + 0\times2^3 + 0\times2^2 + 0\times2^1 + 1\times1$$

and so this same number is written 1001001 in base two.

One often needs to determine the value of a certain digit of a given number: *Is the seventh digit of* 765215 *in base two a* 0 *or a* 1*?* Having a computer command that swiftly determines the k th digit of a number is handy.

Suppose

$$N = a_n\times2^n + a_{n-1}\times2^{n-1} + \cdots + a_k\times2^k + \cdots + a_1\times2 + a_0\times1.$$

To determine if the k th digit a_k is a 0 or a 1, look at

$$\frac{N}{2^k} = 2^{n-k}a_n + \cdots + 2a_{k+1} + a_k + \frac{1}{2}a_{k-1} + \cdots + \frac{1}{2^k}a_0.$$

Since $\frac{1}{2}a_{k-1} + \frac{1}{2^2}a_{k-2} + \cdots + \frac{1}{2^k}a_0$ is no larger than $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}$, a sum smaller than one (think Zeno!), $\frac{N}{2^k}$ is a multiple of two plus a_k plus a fraction no larger than one.

So

$$\operatorname{mod}\left(\frac{N}{2^k}, 2\right) = a_k + \frac{1}{2}a_{k-1} + \cdots + \frac{1}{2^k}a_0$$

and

$$\left\lfloor \operatorname{mod}\left(\frac{N}{2^k}, 2\right) \right\rfloor = a_k.$$

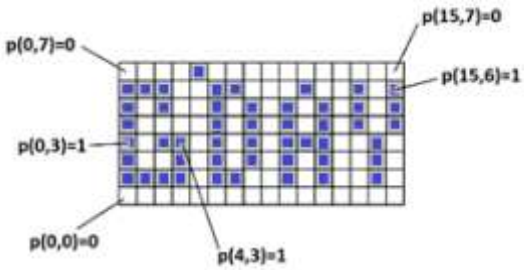
We have a formula for the k th digit in the base-two representation of any integer N.

As a true/false test we see:

> *If* $\left\lfloor \operatorname{mod}\left(\frac{N}{2^k}, 2\right) \right\rfloor > \frac{1}{2}$ *is TRUE, then the* k *th digit of* N *in base two is a* 1. *If it is FALSE, the digit is* 0.

ENCODING A PICTURE

Tupper used a $[0,105]\times[0,16]$ window to draw a picture of his inequality. (It fit nicely?) One can encode any picture, in any sized window, as a number in base two.



In the diagram I have an array of pixels sixteen wide and eight high. Give the bottom left pixel coordinates $(0,0)$ and top right pixel coordinates $(15,7)$. In general, set the coordinates of each pixel as given by a column number (between 0 and 15) and a row number (between 0 and 7). The zeroth column is the left column and the zeroth row is the bottom row.

Let $p(a,b)$ be 1 if the pixel in column a, row b is black; 0 if the pixel is white.

Starting at the bottom left pixel and going up the rows, and then over to the next column to the right and up the rows, and so on, we can write a number N that encodes the picture:

$$\begin{aligned}N = \\ &p(0,0)\times 1 + p(0,1)\times 2 + \cdots + p(0,7)\times 2^7 \\ &+p(1,0)\times 2^8 + p(1,1)\times 2^9 + \cdots + p(1,7)\times 2^{15} \\ &+p(2,0)\times 2^{16} + \cdots + p(2,7)\times 2^{23} \\ &+\cdots \\ &+p(15,0)\times 2^{15\times 8} + \cdots + p(15,7)\times 2^{15\times 8+7}.\end{aligned}$$

We see that $p(a,b)$ is the coefficient of 2^{8a+b}.

If we wish to encode a second picture, one that does not overlap with this one, let's work with pixels with coordinates eight places higher:

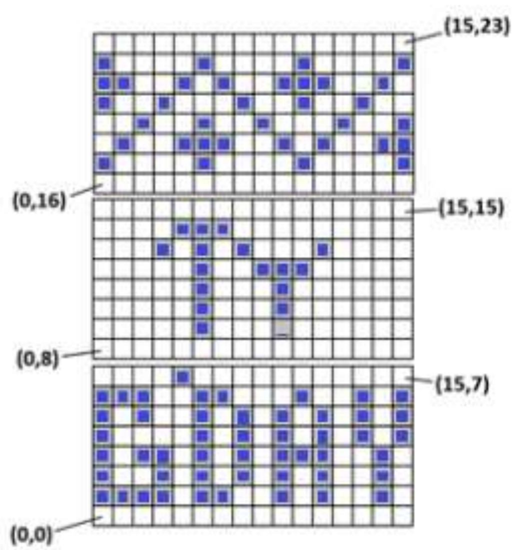
$$\begin{aligned}&p(0,8)\times 1 + p(0,9)\times 2 + \cdots + p(0,15)\times 2^7 \\ &+p(1,8)\times 2^8 + p(1,9)\times 2^9 + \cdots + p(1,15)\times 2^{15} \\ &+p(2,8)\times 2^{16} + \cdots + p(2,15)\times 2^{23} \\ &+\cdots \\ &+p(15,8)\times 2^{15\times 8} + \cdots + p(15,15)\times 2^{15\times 8+7}.\end{aligned}$$

Here $p(a,b)$ is the coefficient of $2^{8a+(b-8)}$. (Notice $b-8$ is the "amount" b is over 8. It is $\text{mod}(b,8)$.)

For a third picture eight places higher again we can use:

$$\begin{aligned}&p(0,16)\times 1 + p(0,17)\times 2 + \cdots + p(0,23)\times 2^7 \\ &+p(1,16)\times 2^8 + p(1,17)\times 2^9 + \cdots + p(1,23)\times 2^{15} \\ &+p(2,16)\times 2^{16} + \cdots + p(2,23)\times 2^{23} \\ &+\cdots \\ &+p(15,16)\times 2^{15\times 8} + \cdots + p(15,23)\times 2^{15\times 8+7}.\end{aligned}$$

Here $p(a,b)$ is the coefficient of $2^{8a+(b-16)}$. (And $b-16$ is the "amount" b is over a multiple of 8. It is $\text{mod}(b,8)$.)



In general, we can have a k th picture starting at $(0,8k)$ encoded by a number

$$p(0,8k)\times 1+p(0,8k+1)\times 2+\cdots+p(0,8k+7)\times 2^7$$
$$+p(1,8k)\times 2^8+p(1,8k+1)\times 2^9+\cdots+p(1,8k+7)\times 2^{15}$$
$$+p(2,8k)\times 2^{16}+\cdots+p(2,8k+7)\times 2^{23}$$
$$+\cdots$$
$$+p(15,8k)\times 2^{15\times 8}+\cdots+p(15,8k+7)\times 2^{15\times 8+7}.$$

Here $p(a,b)$ is the coefficient of $2^{8a+\text{mod}(b,8)}$. (This observation is key in what comes next.)

DELIGHTFUL RECURSIVE QUIRKINESS

Wouldn't it be delightful to have the k th picture starting at $(0,8k)$ be the picture given by the number k itself? This way, as k runs through all the numbers, our work runs through all possible pictures!

[To be clear: Start by writing k in base two as $a_0+a_1\times 2+a_2\times 2^2+\cdots+a_{127}2^{127}$. Then the k th picture is given by $p(0,8k)=a_0$, $p(0,8k+1)=a_1$, …, $p(15,8k+7)=a_{127}$. That is, simply start at the bottom left cell $(0,8k)$ and work upwards and rightwards coloring pixels black according to the 1s that appear in the base two representation of k.]

To work out to which picture a pixel (x,y) lies (here $0\le x\le 15$), write $y=8k+r$ with k as large as possible and $0\le r\le 7$. Then (x,y) belongs to the k th picture.

We see $k=\left\lfloor \frac{y}{8}\right\rfloor$.

Now the color of the pixel at (x,y) is given by $p(x,y)$, which is the coefficient of $2^{8x+\text{mod}(y,8)}$ in the number $k=\left\lfloor \frac{y}{8}\right\rfloor$. This pixel is black if

$$\left\lfloor \text{mod}\left(\frac{\lfloor y/8\rfloor}{2^{8x+\text{mod}(y,8)}},2\right)\right\rfloor>\frac{1}{2},$$

and white otherwise. So if we just plot black all the integer points (x,y) (in the strip $0\le x\le 15$) for which

$$\left\lfloor \text{mod}\left(\frac{\lfloor y/8\rfloor}{2^{8x+\text{mod}(y,8)}},2\right)\right\rfloor>\frac{1}{2},$$

we'd have a plot of all possible pictures one can draw in a $[0,15]\times[0,7]$ array!

HOW TO FIND A PARTICULAR PICTURE

The picture corresponding to a number k, starts at $(0,8k)$. So to find a particular picture …

Encode the picture as a base-two number k by working from the bottom left pixel and working upwards and rightwards, matching black pixels with 1s and white pixels with 0s. Then the picture starts at row $8k$.

TUPPER'S PICTURES

Tupper works with a grid 17 pixels high (instead of 8) and 106 pixels wide (instead of 16). Only slight adjustments are needed in the work we've done.

Plotting black all integer points (x,y) with $0\le x\le 105$ satisfying

$$\left\lfloor \text{mod}\left(\frac{\lfloor y/17\rfloor}{2^{17x+\text{mod}(y,17)}},2\right)\right\rfloor>\frac{1}{2}$$

produces in that vertical strip all possible pictures one can draw in a $[0,105]\times[0,16]$ array of pixels.

The picture corresponding to number k begins on row $17k$.

RESEARCH CORNER

What does Tucker's inequality plot to the right of $x = 105$? To the left of $x = 0$?

What does Tucker's inequality plot in the vertical strip above the final all-black picture?

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