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WHAT COOL MATH!

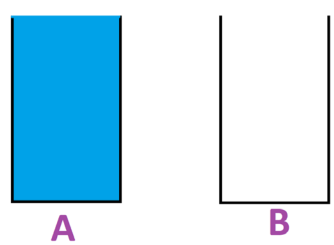
CURIOUS MATHEMATICS FOR FUN AND JOY

APRIL 2020

THIS MONTHS' PUZZLER:

This puzzler is based on a piece recently written for the Exploding Dots website. It's a PUZZLE EXPLAINED BY EXPLODING DOTS. See those puzzles here.

We have two identical containers A and B. Currently A is completely full of water and B is completely empty. We will be pouring water back and forth between the two containers.



In fact, we'll be performing very specific "pouring moves:" at any time we may either pour $\frac{2}{3}$ of the content of container A into container B, or pour $\frac{2}{3}$ of the content of container B into container A.

a) After a finite sequence of pouring moves is it possible to end up with container B precisely one-quarter full?

If the answer is NO, then …

Is it possible to see an amount of water in container B that is so close to one-quarter its volume that the human eye couldn't tell the difference?

b) After a finite sequence of pouring moves is it possible to end up with container B precisely one-third full?

If the answer is NO, then …

Is it possible to see an amount of water in container B that is so close to one-third its volume that the human eye couldn't tell the difference?

c) After a finite sequence of pouring moves is it possible to end up with container B precisely one-half full?

If the answer is NO, then …

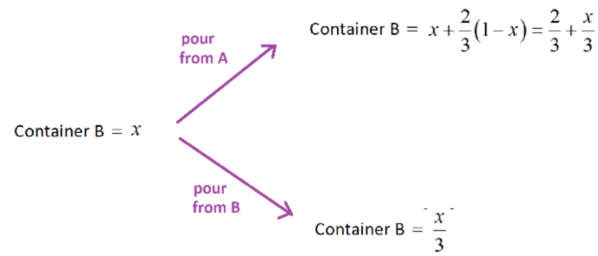
Is it possible to see an amount of water in container B that is so close to one-half its volume that the human eye couldn't tell the difference?

ANALYZING POURING

The fractions $2/3$ and $1/4$ feel "incompatible," as do the fractions $2/3$ and $1/2$, and so we might suspect the answers to parts a) and c) are "no." The answer to part b), on the other hand, might be less clear, intuitively.

There are two possible pouring moves: to pour from container A or to pour from container B.

Let's analyze what happens with each of these moves if we have container B, say, fraction x full. (We want to see $x=\frac{1}{4}$ or $\frac{1}{3}$ or $\frac{1}{2}$.) Container A is then fraction $1-x$ full. And to keep things simple, let's just focus on container B in this analysis.



With both pours we see that we are dividing the given quantity x by three. Perhaps it will be helpful, then, to think of our fractions as expressed in base three, that is, via an $1 \leftarrow 3$ machine: $x = .abcd\ldots$ for some digits a, b, c, d, ….

To be clear:

An ordinary base-ten decimal $.abcd\ldots$ means the quantity

$$\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\frac{d}{10000}+\cdots$$

with a, b, c, d, … any of the digits 0, 1, …, 9.

In base three, a "decimal" $.abcd\ldots$ instead means the quantity

$$\frac{a}{3}+\frac{b}{9}+\frac{c}{27}+\frac{d}{81}+\cdots$$

with a, b, c, d, …. any of the digits 0, 1, 2.

So, if $x = .abcd\ldots$ as a base-three decimal, then

$$\frac{x}{3}=\frac{1}{3}\left(\frac{a}{3}+\frac{b}{9}+\frac{c}{27}+\frac{d}{81}+\cdots\right)$$

$$=\frac{a}{9}+\frac{b}{27}+\frac{c}{81}+\frac{d}{243}+\cdots=.0abcd\ldots$$

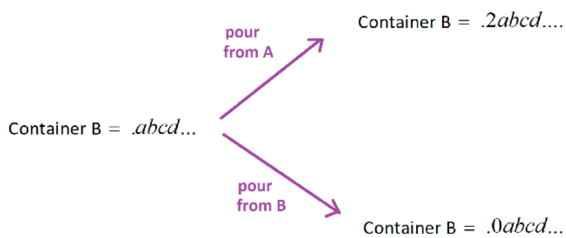
and

$$\frac{2}{3}+\frac{x}{3}$$
$$=\frac{2}{3}+\frac{a}{9}+\frac{b}{27}+\frac{c}{81}+\frac{d}{243}+\cdots=.2abcd\ldots$$
.

From now on, let's assume all our decimal expressions are base-three "decimals."

We have:

If container B is fraction $.abcd\ldots$ *full of water, then a "pour from A" move changes this to fraction* $.2abcd\ldots$ *(insert a 2 into the expression) and a "pour from B" move changes this to fraction* $.0abcd\ldots$ *(insert a 0 into the expression).*



We start with fraction 0 of water in container B. We see that we can thus create the following fractions of water in that container

$$.2=\frac{2}{3}$$

$$.02=\frac{2}{9} \qquad .22=\frac{8}{9}$$

$$.002=\frac{2}{27} \qquad .202=\frac{20}{27}$$

$$.022=\frac{8}{27} \qquad .222=\frac{26}{27}$$

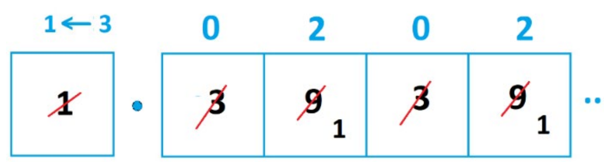
and so on.

SOLVING THE PUZZLES

For part a), we want fraction $\frac{1}{4}$ of water in container B. Can we create it?

To answer that we need to see what $\frac{1}{4}$ is as a base-three decimal.

Let's compute $1 \div 4$ is a $1 \leftarrow 3$ machine.



(Ooh! If we're speaking base three, then $\frac{1}{4}$ should be written $\frac{1}{10}$, yes?)

We see that one-quarter is $.0202020202\ldots.$

With a finite number of pouring moves we can only create fractions that are expressed as finite decimals in base three. As one-quarter is an infinite decimal in this base, we'll never see container B precisely one-quarter full.

But we can create the fractions $.2$ and $.202$ and $.202020202$ and $.20202020202020202020202020202$. That is, we can produce a fraction of water in container B as close to one-quarter as we like!

For part b), we want fraction $\frac{1}{3}$ of water in container B. Can we create it?

As a base-three decimal, this fraction is $.1$, which we cannot create with our pouring moves. But … we can also write this as an infinite decimal using only the digits 0 and 2 as

$$.022222......$$

(Keep unexploding one dot in the $1 \leftarrow 3$ machine.)

We can create the fractions $.02$ and $.022222$ and $.0222222222222222$ via pouring and so can come as close to seeing container B one-third full as we please.

For part c), matters are trickier. The fraction $\frac{1}{2}$ as a base-three decimal is

$$.111111.....$$

(either perform the division $1 \div 2$ in a $1 \leftarrow 3$ machine, or see this is half of $.22222....$ which is 1).

Even with unexplosions, this decimal simply cannot be re-expressed solely using the digits 0 and 2.

Challenge: Prove that any fraction that has a base-three decimal representation using only the digits 0 and 2 is a distance of at least one-sixth of from $\frac{1}{2}$.

We won't ever see container B half full, or even as to close half-full to fool the eye.

THE CANTOR SET

Which numbers in $[0,1]$ have base-three decimal expansions $.abcd...$ which digits solely 0s and 2s?

Looking at the first digit a, the first third of the unit interval represents numbers that have $a = 0$, the middle third those with $a = 1$, and the final third with $a = 2$.

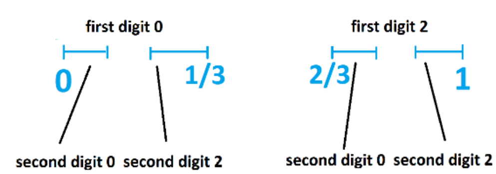
The picture has eliminated all part of the interval corresponding to $a = 1$.

One needs to be a little careful the endpoints, since

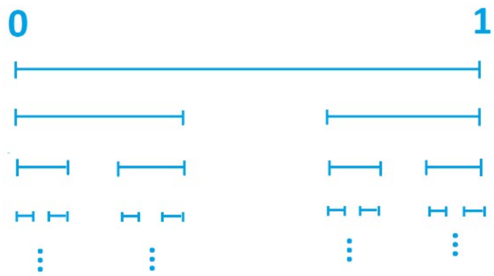
$$0 = .0000...$$
$$\frac{1}{3} = .02222....$$
$$\frac{2}{3} = 0.2$$
$$1 = .22222....$$

all four endpoints can be regarded as "survivors."

Looking at the second digit b, we want to eliminate the surviving parts of the interval that have $b = 1$. This corresponds to removing the middle thirds of the two sections we see. (And one can check that the endpoints can all be regarded as survivors.)



Eliminating all values with $c = 1$ removes middle thirds of sections again (with endpoints surviving), and the same with considering $d = 1$, and so on.



All endpoints that appear along the way survive all the way through. The number $\frac{1}{4}$ is never eliminated during this process either and so also survives all the way through.

The set of points that remain after an infinite process of eliminating (open) middle thirds is a famous construction in mathematics. It is called the Cantor Set and was first described in 1874 by Irish mathematician Henry John Stephen Smith. It has many remarkable properties worth reading about on the internet.

RESEARCH CORNER

Suppose we changed the puzzle so that a "pouring move" consisted of pouring a third of the contents of one container into the other. Which fractions could we see, or come as close to seeing as we like, in conducting pouring moves? (What is a "good" base to work in when analyzing the mathematics?)

How does the puzzle change if we poured $\frac{1}{\sqrt{2}}$ of the contents from one container to the other at each move?

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