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WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

APRIL 2016

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep, joyous, and real mathematical doing I'll happily mention it here.*

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People do math videos!
Check out Marc Chamberlain's https://www.youtube.com/watch?v=bCiQOwP4LrY. Why does $x \mapsto x + \sin(x)$ work?

THIS MONTH'S PUZZLER: It is possible to color the first eight counting numbers each either red or blue so that we never have three distinct integers a, b, and $a+b$ all the same color.

1 2 3 4 5 6 7 8

Can the same task be completed with the first nine counting numbers? What is the smallest N so that every coloring of the numbers 1, 2, 3, …, N either red or blue is sure to have a monochromatic triple $a < b < a+b$?

How does the answer change if we permit generic "triples" with $a = b$? (Now we need each a and $2a$ to be distinct colors too.)

RAMSEY THEORY

Here is a classic result:

If six university students are selected at random, then there is sure to be either three students among the six who are mutual friends or three students who are mutual strangers (or both).

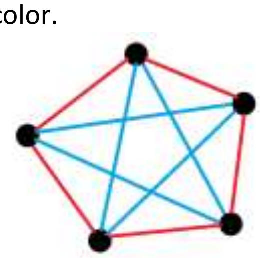
(We are assuming here that friendship is reciprocal: If Albert is friends with Bilbert, then Bilbert is also friends with Albert. Being a stranger is reciprocal too.)

Here's the reasoning: Choose one of the six students, Cuthbert. There are five other students each of which he is either friends with or a stranger to.

Suppose Cuthbert is friends with a majority of these five, that is, friends with at least three of them. (If, instead, he is a stranger to a majority, then switch the words *friend* and *stranger* in what follows.)

Among these three people, if any two are mutual friends, then we have a triple of friends: Cuthbert and those two. If none of those three are friends, then we have found a triple of strangers.

The result is not true for just five people selected at random as seen by this graphic. Here each dot represents a student and a red edge indicates mutual friends and a blue edge mutual strangers. No three people are connected by edges all of the same one color.



In terms of colored diagrams, our party result translates as follows: *Draw six dots on a page and the 15 edges between all possible pairs of dots. It is impossible to color those edges red and blue and avoid a monochromatic triangle.*

To generalize this idea let $R(a,b)$ denote the least number of dots one needs to draw on a page so that if we connect all pairs of dots with either red or blue edges, there is sure to be either a set of a dots with all the edges among them red or a set of b dots with all the edges among those dots blue. (This is assuming that such a least number exists! Maybe no matter how many dots one draws one can always avoid red "cliques" of size a and blue cliques of size b?)

The idea of studying the necessary size of a system to ensure certain sub-substructures exists was first formally explored by British mathematician Frank Ramsey (1903 – 1930). This work is today called Ramsey Theory in his honor.

Our party result reads as $R(3,3)=6$. (Draw six dots and color the edges between then red and blue. Either a red triangle is sure to appear or a blue one.)

It is not hard to see that $R(2,b)=b$.

(If we draw b dots on a page and color the edges, then either one is red and we've found red clique of size 2 or all edges are blue and we have a blue clique of size b. Also, $R(2,b)$ is not $b-1$ or smaller: coloring all the edges between $b-1$ dots blue illustrates this.)

Computing Ramsey numbers is still a very active area of research. Only these few values are currently known.

$R(2,b)=b$ $\quad R(3,4)=9$ $\quad R(4,5)=25$

$R(3,5)=14$

$R(3,6)=18$

$R(3,7)=23$

$R(3,8)=28$

$R(3,9)=36$

(Of course, $R(a,b)=R(b,a)$: just switch colors.)

Generalizing … Set $R(a,b,c)$ as the least number of dots one needs to draw on the page to ensure that, in coloring the edges red, blue and gold, either a clique of a dots with nothing but red edges between them, or a clique of b dots with nothing but blue edges between them, or a clique of c dots with nothing but gold edges between them is sure to appear.

It is known that $R(3,3,3)=17$.

(Draw 17 dots on a page and color each of the 153 edges between them either red, blue, or gold. Then a monochromatic triangle is sure to appear. Also, it is possible to avoid monochromatic triangles with only 16 dots on the page.)

And for full generality set $R(k_1,k_2,...,k_c)$ as the least number of dots one needs to draw on a page so that, in coloring each of the edges between a pair of dots one of c colors, there is sure to be a clique of k_i dots with all the edges between them the i th color, for some i.

Of course, we are assuming that this number exists - that there is a least number of dots that assures a monochromatic structure appears.

Ramsey's Theorem: *Each $R(a,b)$ is indeed a meaningful finite number.*

Let's illustrate why.

The value $R(4,6)$ does not appear on the list of known Ramsey numbers. But we can prove that it is a finite number.

We have, from the list, $R(4,5)=25$ and $R(3,6)=18$. Draw $25+18=43$ dots on the page and color the edges between them red and blue. We shall now reason that either a clique of 4 dots exists with all edges between them red or a clique of 6 dots exists with all edges between them blue. This will establish that $R(4,6)\le 43$.

In our diagram of 43 dots with edges colored, choose one particular dot. Call it Dilbert. Dilbert has some red edges emanating from it connecting it to, say, R other dots. The remaining edges emanating from Dilbert are blue, connecting to B other dots, say. Here $R+B=42$.

Now it can't be that both $R\le 17$ and $B\le 24$. So either R is at least 18 or B is at least 25.

Case $R\ge 18$:

Consider the R dots that connect to Dilbert by red edges. Because $R(3,6)=18$ there is either a red clique of 3 among these R dots or there is blue clique of 6 among them. If there is a red clique of 3, then including Dilbert in the clique (all edges to Dilbert are red) actually means we have a red clique of 4, one of the two structures we are hoping to see for $R(4,6)$. If, on the other hand, there is a blue clique of 6, then we have a blue clique of 6! Either way we have found one of the two things we are looking for.

Case $B \geq 25$:

Consider the B dots that connect to Dilbert via blue edges. Because $R(4,5)=25$, among these B dots there is either a red clique of 4 (one of the possibilities we were hoping for) or a blue clique of 5. In the latter case, since all the edges to Dilbert here are blue, adding Dilbert to the clique of five actually makes a blue clique of 6! Again, we are sure to have at least one of the two structures we were looking for.

In general, one can prove just this way the inequality:

$$R(a,b) \leq R(a,b-1)+R(a-1,b).$$

Then from knowing that Ramsey numbers with smaller indices are finite we can reason that every Ramsey number $R(a,b)$ is finite.

Generalized Ramsey's Theorem: *Each value* $R(k_1,k_2,\ldots,k_c)$ *is finite.*

We have just shown that each of the values $R(a,b)$ for two colorings is a finite number. Let's show how we can use this fact to establish that each of the numbers $R(a,b,c)$ for three colorings must also be finite.

Consider $R(a,b,c)$. We want to show that there is a number N so that if we draw N dots on the page and color the edges either red, blue, or gold, there is sure to be either a red clique of a dots, or a blue clique of b dots, or a gold clique of c dots.

Sometimes when we squint our eyes, red and blue can start to each look purple. So a diagram with edges painted with three colors, red, blue, and gold, can look like a diagram with edges painted just two colors, purple and gold, under squinty eyes.

This gives a way to bring three-colorings back to two-colorings.

Let $n=R(a,b)$. (So any diagram of n dots with edges painted red and blue has either a red clique of a dots or a blue clique of b dots.)

Let $N=R(n,c)$. (So any diagram of N dots with edges painted purple or gold has either a purple clique of n dots or a gold clique of c dots.)

Now draw N dots on the page and color the edges red, blue, and gold. (Remember, we are looking for either a red clique of a dots or a blue clique of b dots or a gold clique of c dots.) Squint your eyes and see only purple and gold. By our choice of N we're either seeing a purple clique of n dots or a gold clique of c dots.

If we're in the latter case, then we've found one of the three things we were hoping to see. If we're in the former case, then we are seeing a purple clique of n dots, which, when we unsquint our eyes, is a set of n dots with red and blue edges between them. But our choice of n was special: it guarantees that either we have a red clique of a dots or a blue clique of b dots. So again, we are seeing one of the three things we were hoping to see.

So $R(a,b,c)$ is finite a number: it is bounded by the number $R(n,c)$ with $n=R(a,b)$.

In general, one reasons this way to show that

$$R(k_1,k_2,k_3,\ldots,k_c) \leq R(n,k_3,\ldots,k_c)$$

with $n=R(k_1,k_2)$. Now knowing that all the three-color Ramsey values are finite, we can use this to argue that all the four-color Ramsey numbers are finite, which leads to all the five-color Ramsey numbers being finite, and so on.

CONNECTIONS TO THE OPENING PUZZLER

Here's a bold claim:

It is impossible to color the counting numbers $1, 2, 3, 4$, …. each one of fifty possible colors and avoid a monochromatic triple a, b, $a+b$. (The generic case $a=b$ is allowed.)

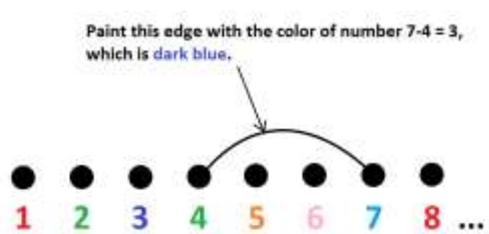
(The number 50 is immaterial here: any finite number of colors will do!)

Here's why.

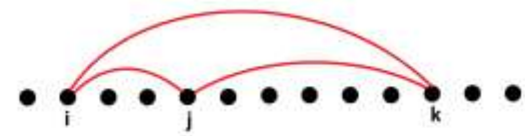
We just proved that the Ramsey value $R(3,3,3,\ldots,3)$, with fifty colors, is a finite value. Let N be its value. So if we draw N dots on a page and color the edges using fifty different colors, then we are sure to find a monochromatic clique of three. That is, we'd find a monochromatic triangle.

Suppose we have colored the counting numbers 1, 2, 3, … each one of fifty colors.

Draw a dot above each of the first N counting numbers and draw an edge between each pair dots. Now color each edge according to the following rule: *Paint the edge connecting the number i to the number j (assume $i<j$ here) with the color of number $j-i$.*



A monochromatic triangle is sure to exist.



From this triangle we have that the color of $j-i$ is the same the color of $k-j$, which is the same as the color of $k-i$.

But observe:

$$(k-j)+(j-i)=(k-i).$$

We have found three numbers a, b, and $a+b$ all the same color.

> **Exercise:** Color each positive integer one color from a given finite set of colors. Must there be a monochromatic triple a, b, ab?

RESEARCH CORNER

1. Let C_k be the smallest value N so that if we color the each of the numbers $1,2,3,\ldots,N$ with one of k colors there is sure to be a monochromatic "triple" $a \le b < a+b$. (We just proved that C_{50} exists and, by easy extension, that each value C_k exists.)

We have $C_1 = 2$ and $C_2 = 5$ (if you did the second part of the opening exercise).

Can you determine any other values of C_k?

2. Let D_k be the smallest value N so that if we color the each of the numbers $1,2,3,\ldots,N$ with one of k colors there is sure to be a monochromatic triple $a < b < a+b$.

We have $D_1 = 3$ and $D_2 = 9$.

Can you adjust the previous proof to establish that the values D_k exist?

3. Explore coloring the positive integers with a finite palette of colors and establishing the existence of a monochromatic quadruple a, b, $a+b$, $a\times b$, with $2<a<b$.

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