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CURIOUS MATHEMATICS FOR FUN AND JOY



October 2017

**THIS MONTH'S PUZZLER:**

*Three points on a circle are chosen at random. What are the chances that all three points lie on one side of some diameter of the circle?*



Global Math Week is this month: Oct 10-17! Hundreds of thousands of teachers and students, from over 90 countries, have signed on to do a joyous, uplifting, delightfully human, and delightfully compelling piece of mathematics together: the story of *Exploding Dots*.

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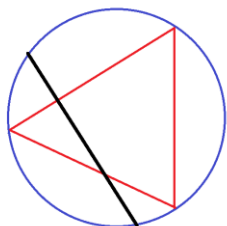
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THE DANGER OF "AT RANDOM"

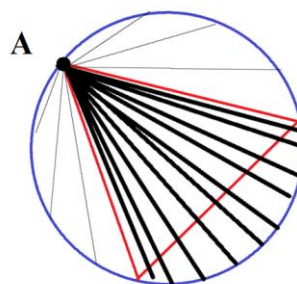
In the late 1800s, French mathematician Joseph Bertrand shocked the mathematics community by illustrating the dangerous ambiguity of the innocent phrase "at random" used in probability problems. He suggested the following challenge.

*An equilateral triangle drawn inside a circle has a certain side length.*

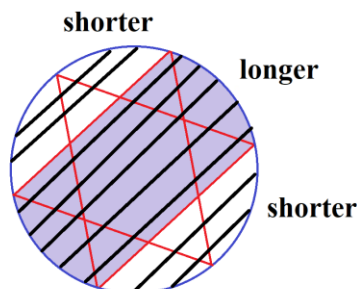
*Choose a circle chord at random. What is the probability that the length of that chord will be longer than the side length of that triangle?*



Answer 1: Choose a point  $A$  on the circle for one endpoint of a chord. Then choose a second point  $B$ . Only if this second point lands in a certain one-third of the perimeter of the circle will chord  $\overline{AB}$  be longer than the side length of the triangle. The probability we seek is thus  $\frac{1}{3}$ .

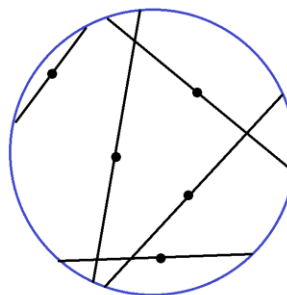


Answer 2: We may group all the chords of a circle into sets of parallel chords. Our chosen chord will be in one of these sets. Moreover, that chord will be longer than the side length of the triangle if it lands in the shaded region shown.

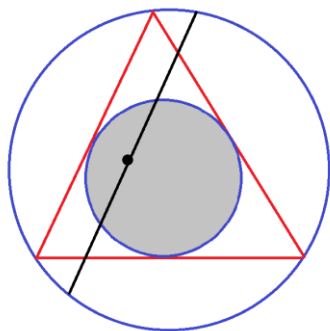


An exercise in basic geometry shows that the width of the shaded region is one-half the width of the circle. Thus the probability we seek is  $\frac{1}{2}$ .

Answer 3: Each point in the circle defines a unique chord—the chord for which that point is a midpoint. So we may as well randomly choose a chord by throwing a dart at the circle. (Well, one point in the circle fails to define a unique chord, namely, the center of the circle. But our chance of hitting the exact center with a dart is zero.)



One sees that if a randomly chosen point lands inside the incircle of the equilateral triangle, then it defines a chord longer than the side length of the triangle.



An exercise in geometry shows that the area of the incircle is one-quarter the area of the entire circle. Thus the probability we seek is  $\frac{1}{4}$ .

Each of these arguments is absolutely mathematically solid and gives a correct final probability value—for the chosen random process it uses.

So we see that one has to be very careful with probability questions that use the phrase “at random” without any indication of which random process one should use.

If one draws a circle on a piece of paper and drops strands of uncooked spaghetti onto it from above, one will find that about  $1/4$  of those strands will cut the circle with a chord longer than the side length of the triangle. If, on the other hand, one draws a circle on the floor and rolls a broom handle from one side of the room to land on the circle, about  $1/2$  of those rolls will give a sufficiently long chord.

We certainly saw this “at random” trouble too in the January 2017 essay on the chances of randomly breaking a stick into three lengths to form a triangle:

[http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay\\_January-2017\\_Sticks-and-Triangles.pdf](http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_January-2017_Sticks-and-Triangles.pdf).

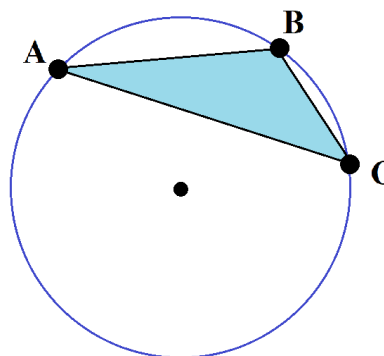
**Question:** Create another valid process of “choosing a circle chord at random” that gives a fourth correct answer to Bertrand’s challenge.

### THE GEOMETRY OF THE OPENING PUZZLER

Consider three points  $A$ ,  $B$ ,  $C$  on a circle. They define a triangle.

Can you see that the following statements are equivalent?

- i)  $A$ ,  $B$ ,  $C$  lie on one side of a diameter.
- ii) Triangle  $ABC$  is obtuse.
- iii) Triangle  $ABC$  does not contain the center of the circle.



If we find an answer to the opening puzzler, that the probability three randomly chosen points lie on one side of diameter is  $P$ , then we will have also shown that

*the chance that those three points define an acute triangle is  $1 - P$*

and that

*the chance that the triangle defined by those points contains the center of the circle is  $1 - P$ .*

We need to find the value of  $P$ .



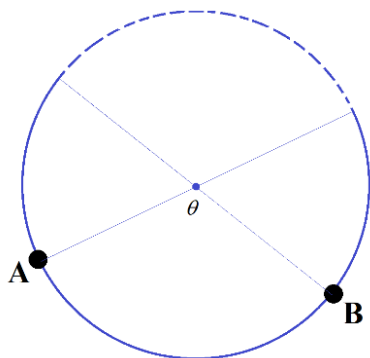
## SOLVING THE OPENING PUZZLER

We realise that the opening puzzler is ambiguous: How are we to choose three points on a circle at random?

Most people would probably agree (but they don't have to!) that a natural approach here is to choose a first point  $A$  on the rim in some (uniformly distributed) random manner—throw a dart onto the perimeter, or instead into the interior of the circle and use the radius it defines to give a point on the rim—and then do the same to find a second point  $B$ , and one more time for a third point  $C$ .

Let's answer the question for this choice of random procedure.

The first two points  $A$  and  $B$  certainly lie on one side of a diameter of the circle. (The chances that they themselves are the endpoints of a diameter is zero.) And they define a region of perimeter we hope point  $C$  will lie so that all three points then lie on the same side of some diameter.



### Hard Calculus Answer:

The two points  $A$  and  $B$  define a central angle  $\theta$  between  $0^\circ$  and  $180^\circ$ . (Would you prefer I worked in radians?) The probability that chosen point  $C$  lands in an acceptable region of perimeter given the locations of

$A$  and  $B$  is  $P_\theta = \frac{360 - \theta}{360} = 1 - \frac{\theta}{360}$ . Now

we need to “sum up” all these probabilities

over all possible range of values of  $\theta$ , and compare this sum with the full range of  $\theta$  values. (Huh?) The probability we seek is

$$\begin{aligned} P &= \frac{1}{180} \int_0^{180} 1 - \frac{\theta}{360} d\theta \\ &= \frac{1}{180} \left( 180 - \frac{1}{2} \cdot \frac{180^2}{360} \right) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4}. \end{aligned}$$

**Warning:** This isn't actually following the random procedure we settled on. First choose a point  $A$ . Then choose a point  $B$ . If we follow a counter-clockwise direction from  $A$  this could lie at any angle  $\theta$  from  $A$  that varies from  $0^\circ$  to  $360^\circ$ . Then chosen point  $C$  has probability  $P_\theta$  of lying in a desired section of perimeter with

$$P_\theta = \begin{cases} 1 - \frac{\theta}{360} & \text{if } 0^\circ \leq \theta < 180^\circ \\ \frac{\theta}{360} & \text{if } 180^\circ \leq \theta < 360^\circ \end{cases}$$

The appropriate integral is

$$\begin{aligned} P &= \frac{1}{360} \int_0^{360} P_\theta d\theta \\ &= \frac{1}{360} \left( \int_0^{180} 1 - \frac{\theta}{360} d\theta + \int_{180}^{360} \frac{\theta}{360} d\theta \right) = \frac{3}{4} \end{aligned}$$

which (phew!) gives the same value.

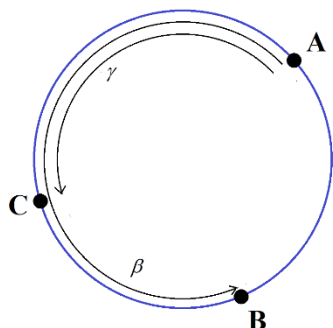
We have:

*Three points on a circle are chosen at random. The chance that those three points define an acute triangle is 25%.*

*Three points on a circle are chosen at random. The chance that the triangle defined by those points contains the center of the circle is 25%.*

Parameter Space Answer:

The point  $A$  will lie somewhere on the circle. The point  $B$  will lie at some angle  $\beta$  counterclockwise from  $A$ , and the point  $C$  some counter clockwise angle  $\gamma$  from  $A$  as well. Here  $0^\circ \leq \beta, \gamma < 360^\circ$ . This picture shows a situation with  $\beta > \gamma$ .



For the case shown with  $\beta > \gamma$ , the three points lie on the same side of some diameter if, and only if, one of the arcs  $AB$ ,  $BC$ , or  $AC$  is more than half the perimeter, that is, if one of these three conditions holds:

$$\gamma > 180^\circ \text{ with } \beta > \gamma$$

$$\beta - \gamma > 180^\circ \text{ with } \beta > \gamma$$

$$360^\circ - \beta > 180^\circ \text{ with } \beta > \gamma.$$

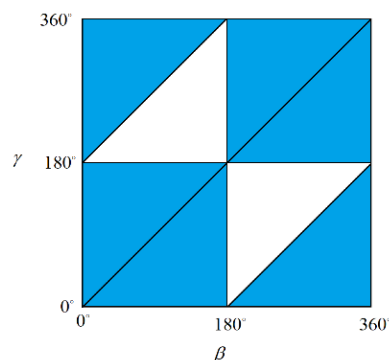
For the case  $\beta < \gamma$  one checks we obtain the conditions:

$$\beta > 180^\circ \text{ with } \gamma > \beta$$

$$\gamma - \beta > 180^\circ \text{ with } \gamma > \beta$$

$$360^\circ - \gamma > 180^\circ \text{ with } \gamma > \beta.$$

This means we need to choose a pair of values  $(\beta, \gamma)$  that lies in one of the shaded regions from all possible locations of such pairs.



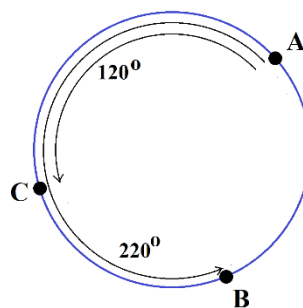
The chances of this happening are

$$P = \frac{3}{4}.$$

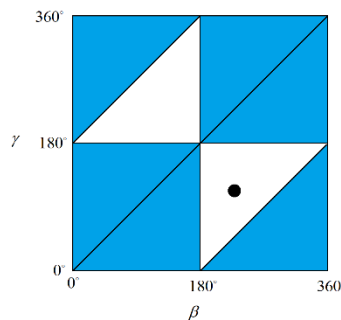
By the way, the points  $(\beta, \gamma)$  that lie in a white region of the parameter-space diagram correspond to acute triangles.

**ON THE WAY TO A SLICK GEOMETRY ANSWER**

Suppose our three chosen points  $A$ ,  $B$ , and  $C$  have the specific values  $\beta = 220^\circ$  and  $\gamma = 120^\circ$ .

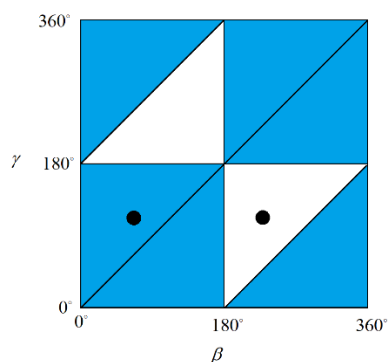
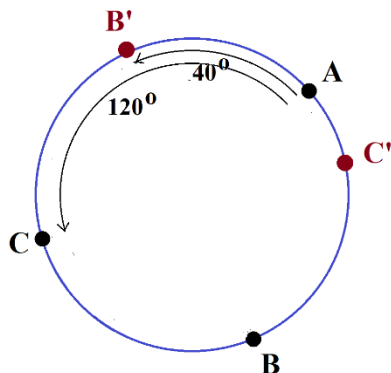


Then  $(\beta, \gamma)$  is the point shown in our parameter space.

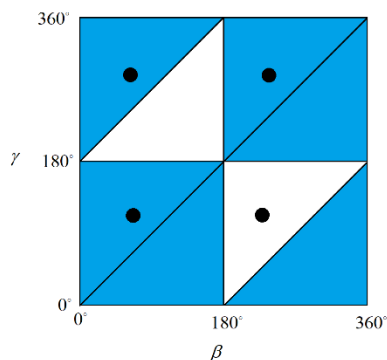


Let  $B'$  and  $C'$  be the points diametrically opposite  $B$  and  $C$ , respectively, on the circle. Then each of the triples  $AB'C$ ,  $ABC'$ , and  $AB'C'$  give us another point in the parameter space.

For example,  $AB'C$  gives us the point  $(40^\circ, 220^\circ)$ .



In the same way,  $ABC'$  gives the point  $(220^\circ, 300^\circ)$ , and  $AB'C'$  the point  $(40^\circ, 300^\circ)$ . (The individual angles each change by  $180^\circ$ .)



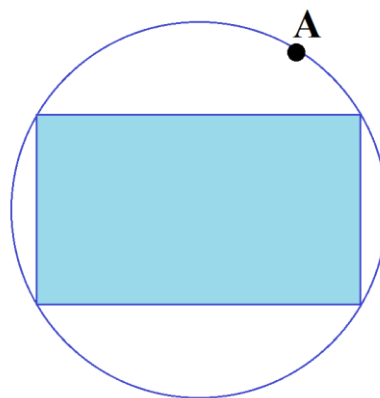
So each triple of points  $ABC$  is a natural representative of a set of four triples:  $ABC$ ,  $AB'C$ ,  $ABC'$ , and  $AB'C'$ , which give a square arrangement of four points in the  $(\beta, \gamma)$  parameter space. It is clear that three out of these four points will lie in a shade region, one will not. Thus the proportion of three chosen points on a

circle that lie given an acute triangle is  $\frac{1}{4}$ ,

and the proportion that don't is  $\frac{3}{4}$ .

As a consequence of our thinking, we have actually proven the following result:

*A rectangle is drawn in a circle and a point  $A$  is drawn on the rim of the circle. There is then precisely one acute triangle with a side of the rectangle as the base of the triangle and the point  $A$  the apex of the triangle.*



**Question:** Is this result "obvious" and easy to prove? If so, then we have a swift answer to the opening puzzler.



## RESEARCH CORNER

1. The three hands of a clock sweep smoothly across a clock face. Assume the hands are all the same length.

Do the tips of the three hands define an acute triangle 25% of the time, literally?

What if the motion of the hands are discrete, each making an instantaneous jolt of movement each second? Again, do the tips define an acute triangle for 25% of the seconds of the day?

2. Four points are chosen at random on the rim of a circle. What are the chances that the convex quadrilateral they define contains the center of the circle? Five points? Six points?  $N$  points?



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