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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

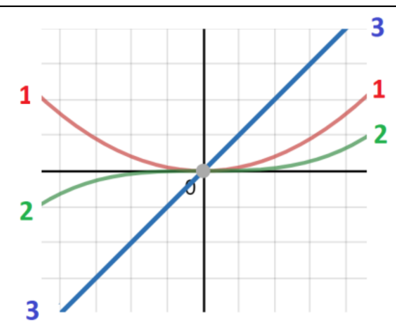
November 2017

THIS MONTH'S PUZZLER:
Polynomial Permutations

Consider the polynomials $P_1(x)=x^2$, $P_2(x)=x$, *and* $p_3(x)=x^3$. *They each pass through the origin. Just to the left of the origin (that is, for small negative values of* x *) we have*

$$P_1(x)>P_2(x)>P_3(x)$$

and just to the right (that is, for small positive values of x *) their order has changed to* $P_3(x)>P_1(x)>P_2(x)$.



Do there exist examples of polynomials $P_1(x)$, $P_2(x)$, *and* $P_3(x)$ *each passing through the origin with* $P_1(x)>P_2(x)>P_3(x)$ *just to the left of the origin, but having instead the order* $P_3(x)>P_2(x)>P_1(x)$ *just to the right?*

How about the order
$P_2(x) > P_3(x) > P_1(x)$ *?*
Is each of the six possible permutations of $\{1,2,3\}$ *a possible reordering of some set of three polynomials in going from left to right of the origin?*

A RESULT OF MAXIM KONTSEVICH

One can indeed find examples of three polynomials passing through the origin with $P_1(x) > P_2(x) > P_3(x)$ just to the left of the origin, and with any desired permutation represented by them just to the right. (Do the opening puzzler!)

We'll loosely say: *Every permutation of* $\{1,2,3\}$ *can be realized as a "polynomial permutation."*

But the same result is not true for four or more polynomials. For example:

There is no set of four polynomials passing through the origin with

$$P_1(x) > P_2(x) > P_3(x) > P_4(x)$$

just to the left of the origin but with

$$P_2(x) > P_4(x) > P_1(x) > P_3(x)$$

just to the right.

That is, the permutation $(2,4,1,3)$ is not a polynomial permutation of $\{1,2,3,4\}$.

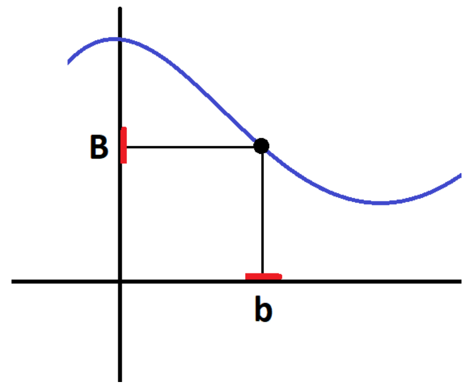
We can prove this, but we need to first gather some results about polynomials.

Polynomial Tidbits

A polynomial is a function P that can be expressed by a rule of the form $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for some fixed real numbers a_i. We'll take it as understood that

The graph of a polynomial is a smooth continuous curve with no surprising "jumps."

By this we mean that if $P(b) = B$ for some input $x = b$, then for all inputs x really close to the value b, all the outputs $P(x)$ will be close to the output value B.



For a polynomial
$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$
to pass through the origin, we must have $a_0 = 0$. So each of the polynomials we wish to consider have the form

$$P(x) = a_kx^k + \cdots + a_nx^n$$

for some $k \ge 1$ with $a_k \ne 0$ (or it is the constant polynomial identically zero).

For a non-zero polynomial of this type, we can say:

If k is odd, then $P(x) = a_kx^k + \cdots + a_nx^n$ crosses the x-axis at the origin. It does not if k is even.

To see this, write $P(x) = x^k\left(a_k + a_{k+1}x + \cdots + a_nx^{n-k}\right)$ and consider the polynomial $q(x) = a_k + a_{k+1}x + \cdots + a_nx^{n-k}$. It satisfies $q(0) = a_k$ and so gives values really close to a_k for x really close to 0. Thus $q(x)$ does not change sign either side of the origin. It follows then that $p(x) = x^k \cdot q(x)$ does change sign if k is odd and doesn't if k is even.

Finally, write $P(x) = a_{k_1} x^{k_1} + \cdots + a_n x^n$ and $Q(x) = b_{k_2} x^{k_2} + \cdots + b_m x^m$ with a_{k_1} and b_{k_2} non-zero. We have

If $|P(x)| < |Q(x)|$ *for all small values of* x *to one side of the origin, then* $k_1 \geq k_2$.

The gist of an argument is this: We have $P(x) = x^{k_1}\left(a_{k_1} + q_1(x)\right) \approx a_{k_1} x^{k_1}$ for small values of x, and $Q(x) \approx a_{k_2} x^{k_2}$. Then $\left|\frac{P(x)}{Q(x)}\right| \approx \left|\frac{a_{k_1}}{a_{k_2}}\right| \cdot |x|^{k_1 - k_2}$. For this to be <1 if x is small, it must be the case that $k_1 \geq k_2$.

Proving the Claim

We're now set to prove that there is no set of four non-zero polynomials passing through the origin with

$$P_1(x) > P_2(x) > P_3(x) > P_4(x)$$

just to the left of the origin but with

$$P_2(x) > P_4(x) > P_1(x) > P_3(x)$$

just to the right.

Let's subtract $P_4(x)$ from each of the polynomials, and set $Q_i(x) = P_i(x) - P_4(x)$ for each i.

Then just to the left of the origin we have polynomials with

$$Q_1(x) > Q_2(x) > Q_3(x) > 0.$$

We want it prove it impossible for them to have the arrangement

$$Q_2(x) > 0 > Q_1(x) > Q_3(x)$$

to the right.

Write these polynomials as

$$Q_1(x) = a_{k_1} x^{k_1} + \cdots$$
$$Q_2(x) = b_{k_2} x^{k_2} + \cdots$$
$$Q_3(x) = c_{k_3} x^{k_3} + \cdots$$

with a_{k_1}, b_{k_2}, c_{k_3} each non-zero.

Suppose we do have $Q_2(x) > 0 > Q_1(x) > Q_3(x)$ to the right of the origin.

Since Q_1 has gone from positive to negative, we must have that k_1 is odd. Similarly, k_3 is odd. And since Q_2 does not change sign, k_2 must be even.

From $Q_1(x) > Q_2(x) > Q_3(x) > 0$ to the left of the origin, we also deduce that $k_3 \geq k_2 \geq k_1$. From $0 > Q_1(x) > Q_3(x)$ to the right of the origin, we deduce that $k_1 \geq k_3$. So we have $k_1 = k_2 = k_3$, with two of the numbers being odd, and the other even! We have a contradiction and so no such polynomials Q_1, Q_2, Q_3, and hence no polynomials P_1, P_2, P_3, P_4 of the types we seek exist.

Question: Did we need to assume that none of the polynomials P_1, P_2, P_3, P_4 was identically zero in this proof?

CHALLENGE: There are 24 permutations of $\{1,2,3,4\}$ and we proved the ordering $P_2(x) > P_4(x) > P_1(x) > P_3(x)$ is unattainable to the right of the origin.

a) Show that $(3,1,4,2)$ is also unattainable as a polynomial permutation.

b) Demonstrate that all remaining 22 permutations can be realized as polynomial permutations.

c) Show that our two "bad" permutations of $\{1,2,3,4\}$ are the only two permutations that fail to map a pair of consecutive numbers to consecutive positions.

Of course the question now arises: Which permutations of $\{1,2,3,\ldots,n\}$ are "polynomial permutations"? That is, for which permutations can we find n polynomials ordered $P_1(x) > P_2(x) > \cdots > P_n(x)$ to the left of the origin, and matching the ordering of the permutation to the right?

For $n=1$, 2, 3 all permutations are polynomial permutations. For $n=4$, all but two of the 24 permutations are polynomial permutations.

If we draw graphs of $n \geq 5$ polynomials through the origin, we'd collect plenty of examples of valid polynomial permutations. By coloring four curves in red, we'd see a valid polynomial permutation of $\{1,2,3,4\}$. Each must be one of the 22 "good" permutations of $\{1,2,3,4\}$.

We have

Any polynomial permutation of $\{1,2,3,\ldots,n\}$ *must avoid the two bad permutations of four elements on all four-element subsets of* $\{1,2,3,\ldots,n\}$.

So in a permutation of $\{1,2,3,\ldots,n\}$ for $n>4$, if we can identify four numbers (a,b,c,d) that reorder to (b,d,a,c) or (c,a,d,b), one of the two problematic permutations for the $n=4$ case, then that permutation must be impossible to attain as a polynomial permutation.

For example, here is a permutation of seven elements

(1,2,3,4,5,6,7)

↓

(1,4,6,7,2,3,5)

We see that is corresponds to seven polynomials with

$P_1 > P_2 > P_3 > P_4 > P_5 > P_6 > P_7$

to the left of the origin, and reordering

$P_1 > P_4 > P_6 > P_7 > P_2 > P_3 > P_5$

to the right.

Focusing on the polynomials P_2, P_4, P_5, P_7 we see they satisfy

$$P_2 > P_4 > P_5 > P_7$$

to the left of the origin and

$$P_4 > P_7 > P_2 > P_5$$

to the right. No two consecutive terms are being permuted to consecutive positions, so this is a bad permutation on four elements. Our permutation on seven elements thus fails to be a polynomial permutation.

Here's the general result.

KONTSEVICH'S THEOREM: *A permutation is indeed a polynomial permutation if it avoids the two bad permutations on four elements on all four-element subsets.*

Proving that each and every such representation is realizable with polynomials is quite the task! This is the subject of Étienne Ghys's spectacular book "A singular mathematical promenade" (2017) downlable from the web for free. Check it out!

RESEARCH CORNER

Can you prove Kontsevich's theorem without looking at the cited text?

Let $a(n)$ be the number of polynomial permutations of $\{1,2,3,\ldots,n\}$. (We have, for instance, $a(4)=22$.) Is there an easy way to compute $a(5)$ and $a(6)$? Is there a there a general formula for $a(n)$?

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GLOBALMATHPROJECT

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All was grassroots. All was volunteer. And all was propelled by a stunning community of teachers, coaches, leaders, enthusiasts – you! -- simply wanting to bring joyous mathematics to the world.

THANK YOU!

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