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WHAT HO! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

May 2018

THIS MONTH'S PUZZLER:

We have that six equals $(-2)^4+(-2)^3+(-2)^1$ and negative six equals $(-2)^3+(-2)^2+(-2)^1$.

Is it true that each and every integer, positive or negative, is a sum of distinct powers of negative two?
Can the same one integer be represented as a sum of distinct powers of negative twos in more than one way?

ABOUT THIS ESSAY

This material appears on the website www.gdaymath.com as Experience 11 of the *Exploding Dots* material. I share it here as I think the work presented here is worthy of wide dissemination. It shows that the power of *Exploding Dots* is an age-old idea and that Scottish mathematician John Napier took the power of a $1 \leftarrow 2$ machine to great heights some four hundred years ago.

The great Martin Gardner wrote about this work too in his article "Napier's Chessboard Abacus." It appears as chapter 8 in *Knotted*

Doughnuts and Other Mathematical Entertainments (W.H.Freeman and Company, 1986). But the work there is not at all framed in terms of a $1 \leftarrow 2$ machine. I think seeing that content framed in the context of *Exploding Dots* is well worth the while.

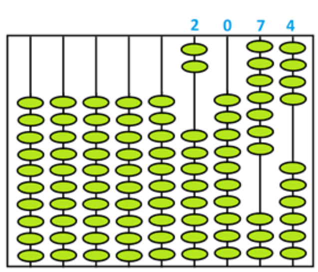
This essay presumes very basic familiarity with the $1 \leftarrow 2$ machine. Take a few moments to watch the first video or two of Experience 1 in the Exploding Dots course mentioned about and you will be all set.

Enjoy!

NAPIER'S CHECKEBOARD

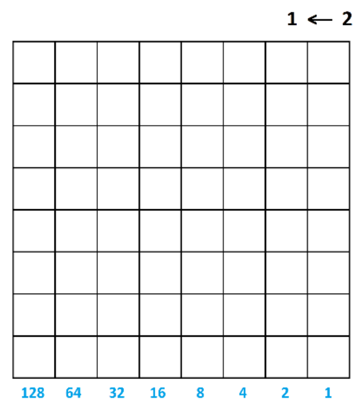
The concept of *Exploding Dots* has been around for many centuries, though not necessarily visualized as dots in boxes (and certainly not as exploding dots).

The ancient counting and arithmetic device, an *abacus*, is simply a $1 \leftarrow 10$ machine. Its simplest version is just a series of rods held in a frame with each rod holding ten beads. One slides beads up rods to represent numbers and, in performing calculations, whenever ten beads reach the top of one rod, one slides them down (they "explode") and raises one bead up on the rod one place to their left in their stead.



Comment: A more modern abacus has a cross bar with five beads on each rod below the bar and two beads above it, with each of those two beads representing a group of five. One slides beads to touch the cross bar. Thus "8," for example, is represented on a rod as three beads touching the cross bar from below and one bead touching the cross bar from above. This version of the abacus is a $1 \leftarrow 10$ machine that has a special dot (a blue dot, perhaps) that represents five dots in a box.

Four centuries ago, Scottish mathematician John Napier (1550 – 1617), best known for his invention of logarithms, actually discovered and worked with a $1 \leftarrow 2$ machine, but he found it useful to stack rows of boxes on top of one another to make a grid of squares, with each row being its own $1 \leftarrow 2$ machine.



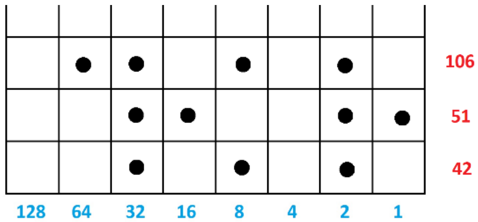
He suggested using a physical copy this grid, a wooden board or square sheet of cloth marked into squares, and beads or counters.

With this board, Napier showed the world how to add, subtract, multiply and divide numbers. He also felt it was useful for computing integer square roots of numbers!

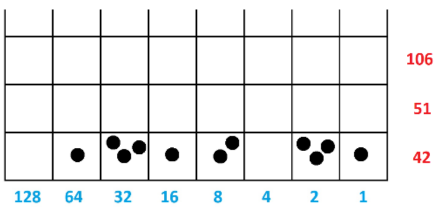
Read on to see how.

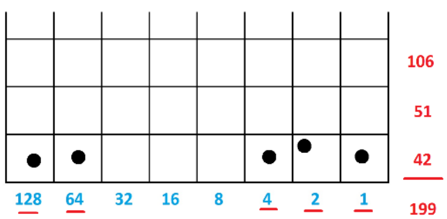
Addition

To add three numbers, say, 106, 53, and 42, represent each number on its own row of the board using counters as dots in a $1 \leftarrow 2$ machine. (Of course, Napier did not use our language of Exploding Dots and their machines, but it is clear how our language translates to actions to do with physical counters on the board.)



Then slide all the dots down to the bottom row and perform the usual $1 \leftarrow 2$ explosion rule to read off the final answer.

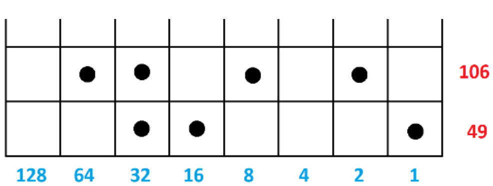




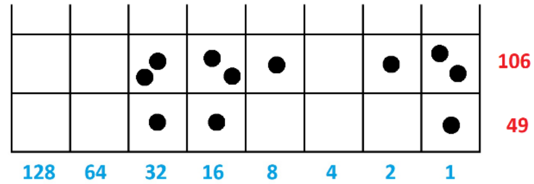
Subtraction

Napier did not introduce the notion of an antidot, but suggested performing subtraction this way instead.

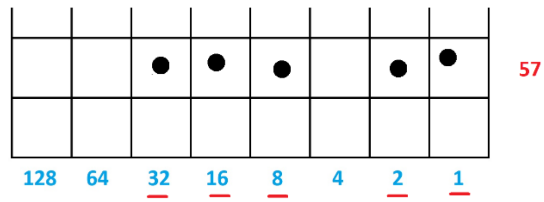
To compute $106 - 49$, say, represent the larger number on the second row of the board and the smaller number on the bottom row.



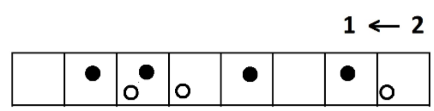
Starting at the left of the second row, perform unexplosions so that each dot in the bottom row has at least one dot above it.



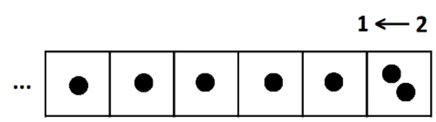
Now subtract dots from the second row, one for each dot that sits on the first row. We see the answer 57 appear.



Question: *The picture below shows how we performed subtraction in the* $1 \leftarrow 2$ *machine using antidots. Can you see a correlation of the two approaches?*

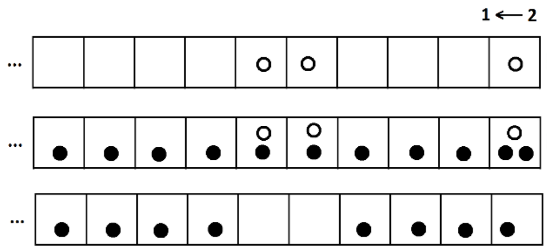


Question: *Consider a* $1 \leftarrow 2$ *machine fully loaded as shown.*



Do you see if you perform all the explosions, all the dots disappear? This shows that, in some sense, the infinitely long base-two number $\cdots 111112$ *represents the number zero. (See the website chapter on* Some Unusual Mathematics for Unusual Numbers *for more on this.)*

This means we can add $\cdots 11112$ *to a picture of a negative number and not change the number. For example, we see* *that another representation of* -49 *in a* $1 \leftarrow 2$ *machine is* $\cdots 11111001111$.



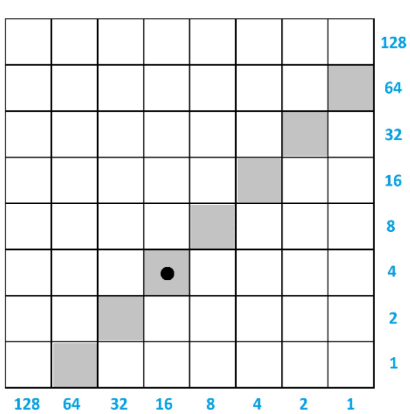
Thus every negative number can be presented in a $1 \leftarrow 2$ *machine without the use of any antidots. (The trade-off is that one must then use an infinite number of dots!)*

Compute $106-49$ *in Napier's checkerboard again but this time thinking of it as an addition problem,* $106+(-49)$*, that can be presented on the board using only dots.*

Multiplication

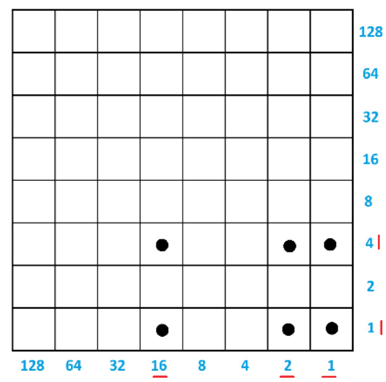
This is where Napier's brilliance starts to shine.

To perform multiplication, Napier suggested viewing the columns of the checkerboard as their own $1 \leftarrow 2$ machines! This way, each dot in a box represents a product. For example, in this picture the dot has value the product $16 \times 4 = 64$.

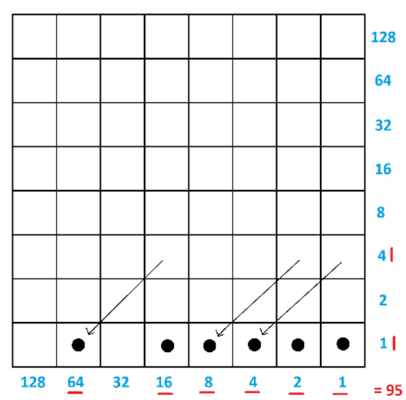


What is lovely here is that dots in the same diagonal have the same product value: $64 \times 1 = 32 \times 2 = 16 \times 4 = \cdots = 1 \times 64$. So in addition to doing $1 \leftarrow 2$ explosions horizontally and vertically, we can also slide dots diagonally and not change the total value represented by dots on the board.

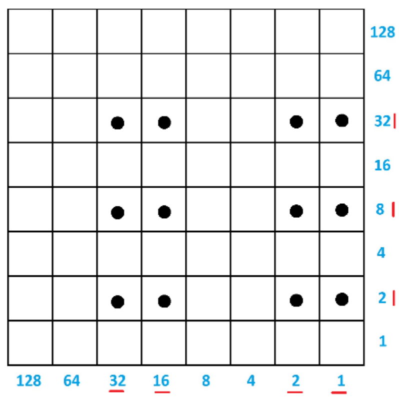
Here's a picture of one copy of 19 plus four copies of 19, that is, here is a picture of 19×5.



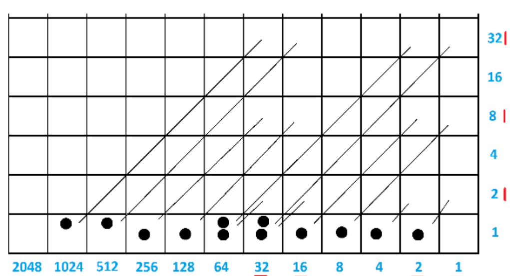
Slide each dot diagonally downward to the bottom row: this does not change the total value of the dots in the picture. The answer 95 appears.



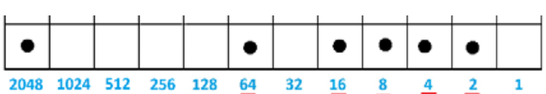
More complicated multiplication problems will likely require using a larger grid and performing some explosions. For example, here is a picture of 51×42.



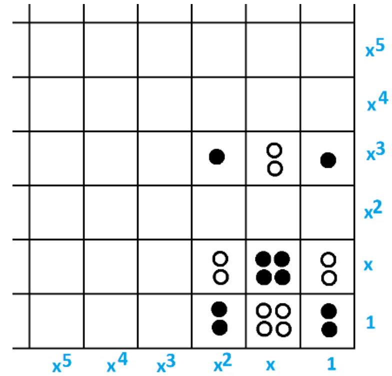
Sliding gives this picture



and the bottom row explodes to reveal the answer 2142.



Question: *One can do polynomial multiplication with the checkerboard too! (One needs two different colored counters: one for dots and one for antidots.) Do you see how this picture represents* $\left(x^2-2x+1\right)\left(x^3-2x+2\right)$*? Do you see how to get the answer* $x^5-2x^4-x^3+6x^2-6x+2$ *from it?*

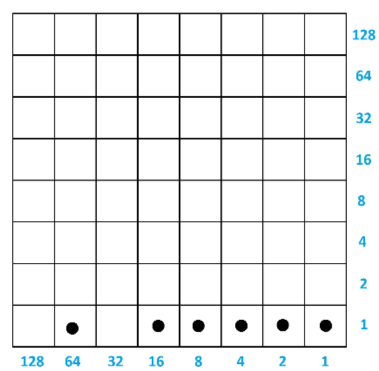


Question:*How would you display the product* $(1-x)\left(1+x+x^2+x^3+x^4+\cdots\right)$*?*

What answer does it give?

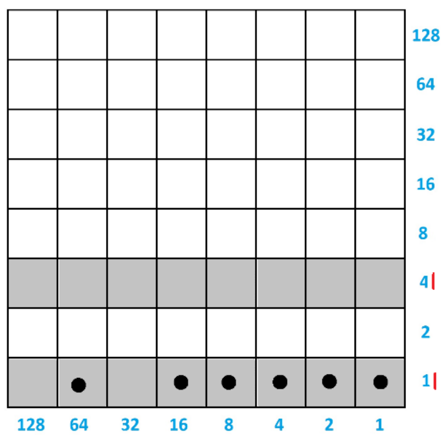
Division

Earlier we computed 19×5 and got this picture for the answer 95.

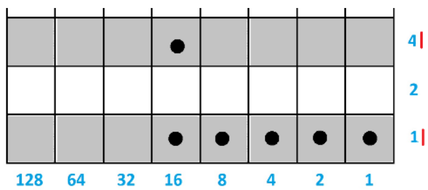


If we were given this picture of 95 first and was told that it came from a multiplication problem with one of the factors being 5, could we deduce what the other factor must have been? That is, can we use the picture to compute $95 \div 5$?

Since $5 = 4+1$ we will need to slide counters on this picture so that two copies of the same pattern appear in the shaded two rows.

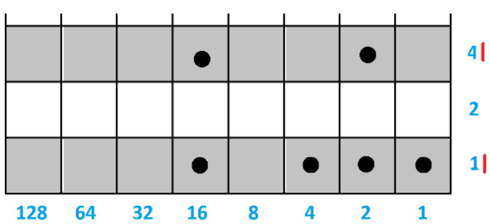


Slide the leftmost dot up to the top shaded row and we see it "completes" the 16 column. Let's not touch the counters in that column ever again.

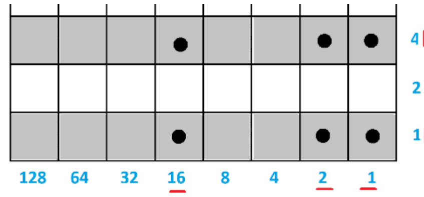


We are now left with a smaller division problem: dividing $8+4+2+1$ (that is, 15) by 5.

Slide its leftmost dot up to the top shaded row. This completes the 2s column and let's never touch the counters in that column again.



This leaves us with a smaller division problem to contend with: $4+1$ divided by 5. Slide its leftmost dot up to the top shaded row to complete the 1s column.



We see that we have now created the picture of 19×5 and so $95\div 5=19$.

This loosely illustrates the general principle for doing division on Napier's checkerboard:

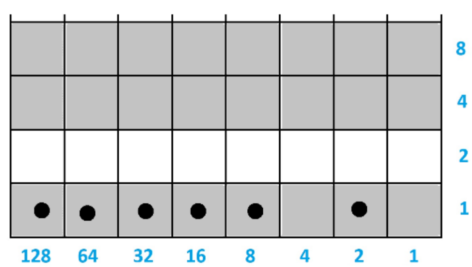
Represent the dividend by dots in the bottom row and the divisor by shaded rows.

Slide the leftmost dot to the top shaded row.

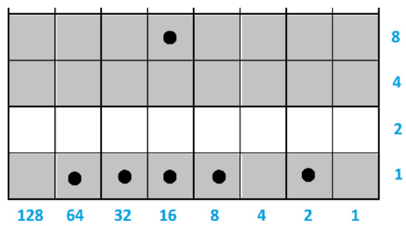
Complete the leftmost column of dots possible in some way you can (you might need to unexplode some dots) and when done never touch those dots again. What is left is a smaller division problem and repeat this procedure for the leftmost dot of that problem.

The procedure described here is loose as our computation $95\div 5=19$ ran into no difficulties.

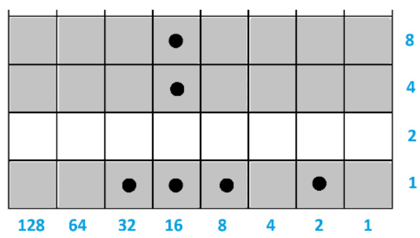
Let's try $250\div 13$ for something more involved. Here's its setup.



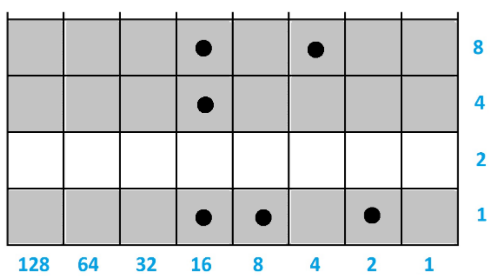
Slide the leftmost dot to the highest shaded row. Doing so shows we need to work with the 16s column, but it is not complete.



We can complete it by sliding the current leftmost dot into that column. (That's convenient!)



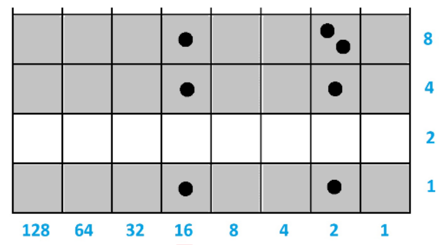
Now we have a smaller division problem to work on. Slide the leftmost dot up to the highest shaded row.



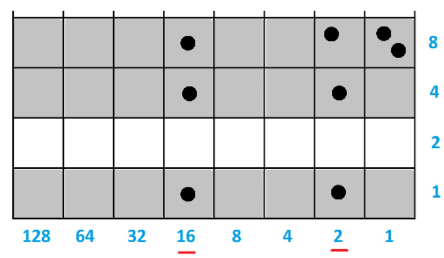
What's the leftmost column we can complete right now without ever touching those dost of the 16s column? We see that there is no means complete the 8s column. (What dot can we slide into its top?)

There is no means to complete the 4s column either. (How do we slide a dot into that 4×4 cell?)

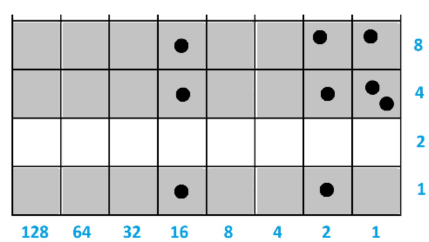
So let's work on the 2s column. I can see by sliding the dot in the 8s column and performing a (horizontal) unexplosion from the 4s column we can fill up the 2s column.

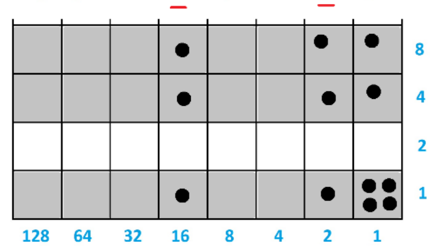


The 2s column is a bit overloaded. Let's unexplode one of the dots the top pair (horizontally).



All the action is now left in the 1s column. What can we do to make that column complete? (Remember, dots in completed columns are never to be touched again.) Let's unexplode downwards a number of times.





This does complete the 1s column, but with three ones too many.

If we had three less dots— 247 instead of 250 —then we would have, right now, a picture of 19×13 showing that

$247 \div 13 = 19$. So it must be then that $250 \div 13$ has a remainder of three and so

$$250 \div 13 = 19 + \frac{3}{13}.$$

Question: *Compute* $256 \div 10$ *via Napier's method.*

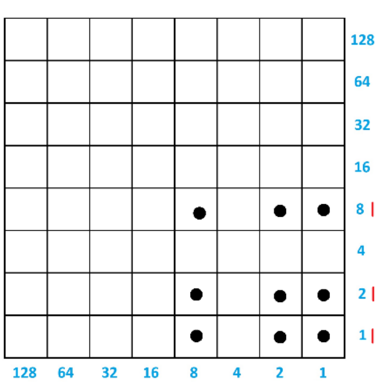
Question: *Is it possible to do polynomial division with Napier's checkerboard? (Can one compute* $\frac{1}{1-x}$*?)*

Wild Explorations

Exploration 1: Squares and Square Roots

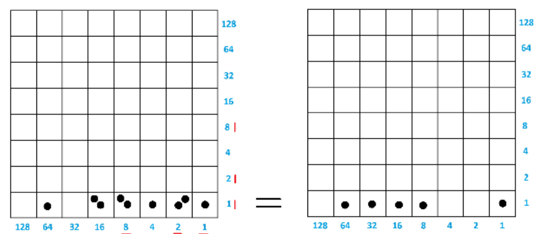
Napier claimed that his checkerboard is also capable of computing integer square root approximations to numbers. For example, his checkerboard can show that $145 = 12^2 + 1$ with 12 being the integer part of $\sqrt{145}$, and that $1000 = 31^2 + 39$ with 31 being the integer part of $\sqrt{1000}$, and so on.

To get a sense of how one might do this, consider first this picture of 11×11 to give the square number 121.



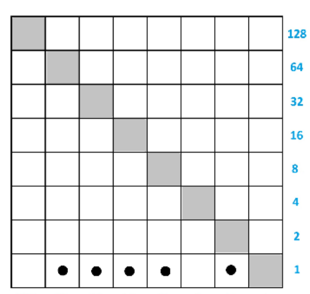
Notice the symmetry about the north-west diagonal: the picture has a pattern of dots on the bottom row, the same pattern of dots in the right column, and the same pattern appears on the diagonal too. Also, each dot in the interior of the picture sits above a dot in the bottom row and to the left of a dot in the rightmost column. All pictures of numbers squared will have such symmetry.

Sliding the dots downwards reveals 11×11 as 121.



Napier claimed that you can reverse this process and reconstruct the symmetric pattern of dots to see that 121 is eleven squared.

Can you indeed slide the dots that represent 121 on the bottom row diagonally upwards (or do some unexplosions and slide unexploded dots upwards) to recreate a picture of 11×11? The key is to focus on the northwest diagonal. Can you do this in a systematic way that you could explain your steps easily to a friend?



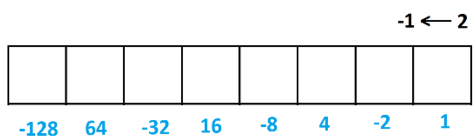
Use your method with the number 145 represented on the bottom row. Can you construct the picture of 12×12 (without knowing that one is looking for 12 to begin with) along with one extra dot in the 1×1 cell?

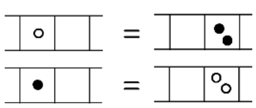
Use Napier's checkerboard to show that $1000 = 31^2 + 39$. (Again, presume you don't know that you are looking for the number 31.)

Exploration 2: Negative Numbers

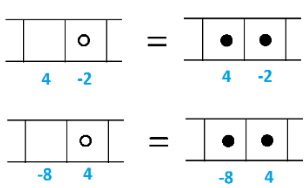
In his book *The Art of Computer Programming, Vol. 2.* (1969) Donald Knuth introduces the *negabinary system*. Here every integer, positive and negative, is represented as a sum of powers of -2 using the coefficients 0 and 1.

In the language of Exploding Dots, negabinary is a $-1 \leftarrow 2$ machine where two dots in one box explode to be replaced by one antidot, one box to the left, and similarly two antidots in a box explode to be replaced by one dot, one box to the left.



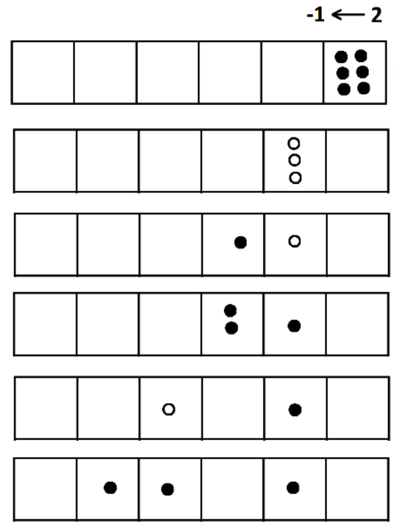


But to avoid the appearance of antidots in the representations of numbers we observe that one antidot in a box is equivalent two dots, one in the original box and one, one place to the left.

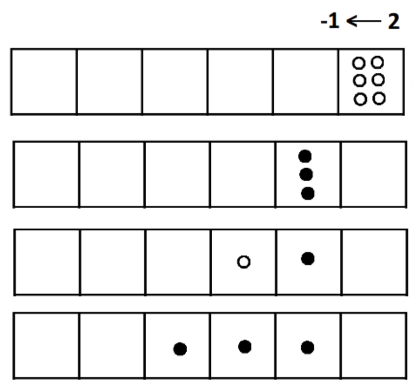


etc.

Placing six dots in the $-1 \leftarrow 2$ machine and using this convention to avoid antidots gives the negabinary code 11010 for six.



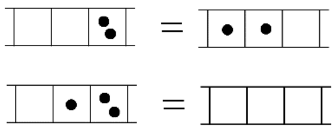
The code for -6 in this machine is 1110.



Work out the negabinary codes of all the integers from -10 to 10. Are there any patterns to be noticed and explained? (For example, which numbers give codes with an even number of digits? Which with an odd number of digits? Can you find a rule for divisibility by two? By three? Which numbers give palindromic codes?)

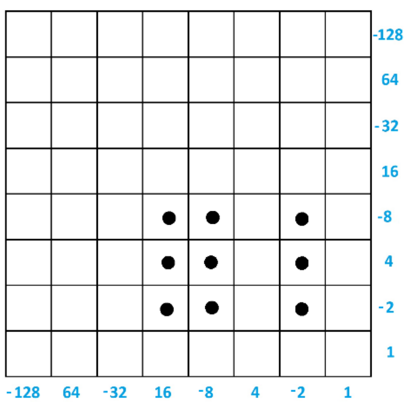
The $-1 \leftarrow 2$ machine shows that it is possible to represent each integer, positive or negative, in base -2 using only the digits 0 and 1 in at least one way. Prove that no integer can have two different base -2 representations using the digits 0 and 1 .

Napier wasn't using two differently colored counters in his work, one for dots and one for antidots. To follow suit, note that we can rephrase the rules of the $-1 \leftarrow 2$ machine solely in terms of dots.



Knuth suggests using Napier's checkerboard with columns and rows labeled with values the powers of -2 , representing numbers with counters in negabinary, and using the above two rules on the board (along with diagonal sliding) to manipulate pictures and thus do calculations.

For example, here is a picture of $6 \times (-6)$. Do you see how to obtain the answer -36 from it?



Compute $6+(-7)$ and $6-(-7)$ and $6\times(-7)$ in this negabinary checkerboard.

The number negative one has code 11 in negabinary. So to change the sign of a number in negabinary we can multiply that number by 11, that is, by $10+1$. Now multiplying a number by 1 does not change the code of the number and multiplying by 10 shifts all the digits of code one place to the left. So to change the sign of a number in negabinary we can write down the code for the number, write a same code with a zero addended, and add those two codes.

Compute $6-(-7)$ as an addition problem of three terms: the code for 6, the code for -7, and the code for -7 with a zero addended, all added together. Did you get the same answer as you did in part c)?

Is there a way to perform divisions on this board too? (Try $38 \div (-13)$.)

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