CURRICULUM INSPIRATIONS: www.maa.org/ci

Innovative Online Courses: www.gdaymath.com
Tanton Tidbits: www.jamestanton.com

WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

March 2018

THIS MONTH'S PUZZLER:

A triangle has vertices $A=(2,3)$, $B=(4,1)$, and $C=(6,2)$. What are the coordinates of its circumcenter, incenter, orthocenter, and centroid?

This essay is motivated by my recent realization that I didn't know how to find the coordinates of the incenter of a triangle. Like every good scholar I Googled it, found a formula for computing its coordinates, but, alas, no satisfactory explanation of the formula. This essay is my attempt to give a full explanation of it.

THE ARITHMETIC OF POINTS

If $A=(a_1,a_2)$ and $B=(b_1,b_2)$ are points in the plane, then it seems reasonable to declare $A+B$ to be the point (a_1+b_1,a_2+b_2) in the plane. Similarly, we can set $2A-B$ to be the point $(2a_1-b_1,2a_2-b_2)$, $\frac{A+B}{3}$ to be $\left(\frac{a_1+b_1}{3},\frac{a_2+b_2}{3}\right)$, and so on. That is, we can declare any linear combination of points in the plane to be the point with x- and y- coordinates given by the same linear combination. The challenge is to

© James Tanton 2018

develop a geometric intuition to these arithmetic constructs.

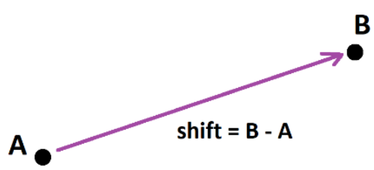
One key piece of this comes from thinking about the motion needed to move between points. For instance, to move from point $A=(3,5)$ to point $B=(9,12)$, we must move $9-3=6$ units to the right and $12-5=7$ units upwards. Writing B as

$$B=\left(3+(9-3),5+(12-5)\right)$$

shows explicitly how the each of the coordinates of A are modified by adding the "shift" from A to B. Using our notation of arithmetic of points this reads

$$B=A+(B-A)$$

which looks correct to our algebra eyes too!



If we conduct only half the shift from A to B we should reach the midpoint of $\overline{AB}$. And indeed we do! Using abstract coordinates this time, if $A=(a_1,a_2)$ and $B=(b_1,b_2)$, then

$$\begin{aligned}&A+\frac{1}{2}(B-A)\\&=\left(a_1+\frac{1}{2}(b_1-a_1),a_2+\frac{1}{2}(b_2-a_2)\right)\\&=\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right).\end{aligned}$$

This is the standard midpoint formula.

We can obtain a formula for the three-sevenths point along the line segment this way too!

$$\begin{aligned}A+\frac{3}{7}(B-A)&=\frac{4}{7}A+\frac{3}{7}B\\&=\left(\frac{4}{7}a_1+\frac{3}{7}b_1,\frac{4}{7}a_2+\frac{3}{7}b_2\right)\end{aligned}$$

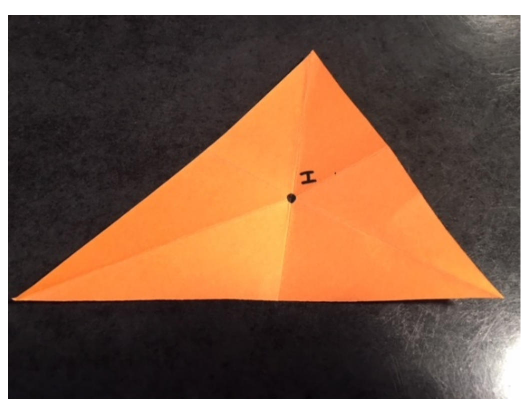
In fact, we can reach any point on the line $\overleftrightarrow{AB}$ by finding the suitable value t and computing $A+t(B-A)=(1-t)A+tB$. (If $0<t<1$, then $A+t(B-A)$ Is a point on the line segment $\overline{AB}$, with $t=0$ giving A and $t=1$ giving B. Points with $t>1$ and $t<0$ lie on the line beyond the line segment.)

Understanding that "$A+shift$" gives us a new point in the plane is all the geometric intuition we need in what follows.

THE INCENTER OF A TRIANGLE

One learns in a geometry course that the three angle bisectors of a triangle meet at a common point I. (Points on an angle bisector are equidistant from two sides of a triangle. Thus the point where just two angle bisectors meet must be equidistant from all three sides, and so must lie on the third angle bisector as well.)

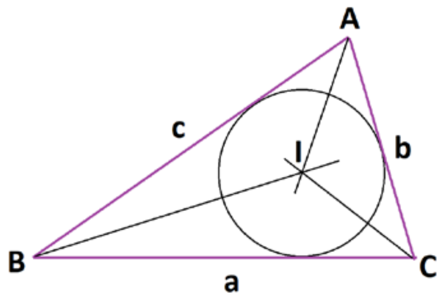
One can fold the angle bisectors of a paper triangle and see the incenter appear.



As this point I is equidistant from all three sides of the triangle, it is possible to draw a

© James Tanton 2018

circle inside the triangle with center I and tangent to all three sides. For this reason I is called the *incenter* of the triangle.



If the triangle has vertices A, B, C with sides opposite those vertices of lengths, respectively, a , b , c, then there is a beautiful formula for the incenter of the triangle. It's

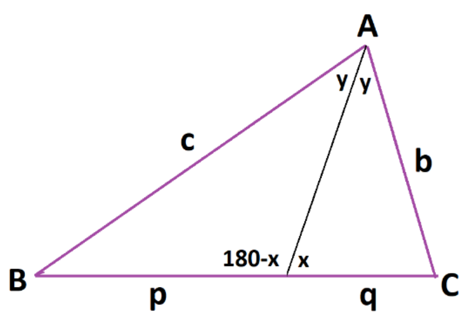
$$I = \frac{a}{a+b+c}A + \frac{b}{a+b+c}B + \frac{c}{a+b+c}C$$

The Angle Bisector Theorem

To prove this formula we need one fact about angle bisectors.

An angle bisector of a triangle cuts the opposite side it meets in the same ratio as its two adjacent sides.

For the diagram shown we specifically have $p = \frac{ca}{b+c}$ and $q = \frac{ba}{b+c}$.



To prove this, label the angles as shown.

The law of sines gives $\frac{b}{\sin(x)} = \frac{q}{\sin(y)}$

and $\frac{c}{\sin(180-x)} = \frac{p}{\sin(y)}$. Noting that $\sin(180-x) = \sin(x)$ we can then see that both $\frac{b}{q}$ and $\frac{c}{p}$ equal $\frac{\sin(x)}{\sin(y)}$. Thus we have that $\frac{p}{q} = \frac{c}{b}$.

From this and the fact that $p+q=a$ we get $p = \frac{c}{b}q = \frac{c}{b}(a-p)$ yielding

$$p = \frac{ac}{b+c}.$$

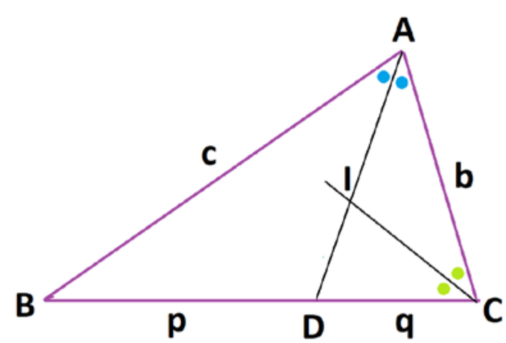
Similarly

$$q = \frac{ab}{b+c}.$$

The Coordinates of I .

Let D be the point where the angle bisector from A meets the opposite side. This is the fraction $\frac{p}{p+q}$ along the line segment $\overline{BC}$, so

$$\begin{aligned} D &= B + \frac{p}{p+q}(C-B) \\ &= \left(1-\frac{p}{p+q}\right)B + \frac{p}{p+q}C. \\ &= \frac{q}{p+q}B + \frac{p}{p+q}C \end{aligned}$$



Now $\frac{p}{p+q} = \frac{p}{a} = \frac{c}{b+c}$ and $\frac{q}{p+q} = \frac{b}{b+c}$, so we have

© James Tanton 2018

$$D = \frac{b}{b+c}B + \frac{c}{b+c}C\ .$$

By exactly the same work applied to the angle bisector in triangle ADC we see

$$I = \frac{q}{b+q}A + \frac{b}{b+q}D.$$

Thus

$$I = \frac{q}{b+q}A + \frac{b}{b+q}\left(\frac{b}{b+c}B + \frac{c}{b+c}C\right)$$

Using $q = \dfrac{ab}{b+c}$ we see that

$$\frac{q}{b+q} = \frac{ab}{b(b+c)+ab} = \frac{a}{a+b+c}$$

$$\frac{b^2}{(b+q)(b+c)} = \frac{b^2}{(b+c)b+ab} = \frac{b}{a+b+c}$$

$$\frac{bc}{(b+q)(b+c)} = \frac{bc}{(b+c)b+ab} = \frac{c}{a+b+c}$$

and so

$$I = \frac{aA+bB+cC}{a+b+c}$$

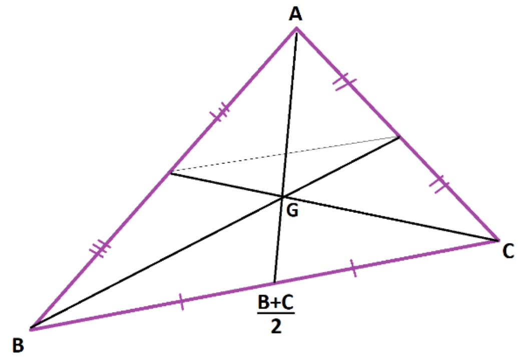
as claimed.

For the triangle of the opening puzzle we have

$$I = \left(\frac{2\sqrt{5}+4\sqrt{17}+6\sqrt{8}}{\sqrt{5}+\sqrt{17}+\sqrt{8}}, \frac{3\sqrt{5}+\sqrt{17}+2\sqrt{8}}{\sqrt{5}+\sqrt{17}+\sqrt{8}}\right)$$

THE CENTROID

The three medians of a triangle meet at a common point G called the *centroid* of the triangle.



An exercise in chasing similar triangles shows that any two medians cut each other at a point one-third along each median, which then allows one to deduce that all three medians do meet at a common point.

One can fold the medians of a paper triangle (first pinch the locations of the midpoints of each side) and see the centroid appear.

If a triangle has vertices A, B, C, then its centroid G can be "reached" by moving from the midpoint $\dfrac{B+C}{2}$ one-third of the way along the median from A. We have

$$G = \frac{B+C}{2} + \frac{1}{3}\left(A - \frac{B+C}{2}\right)$$ giving the lovely formula

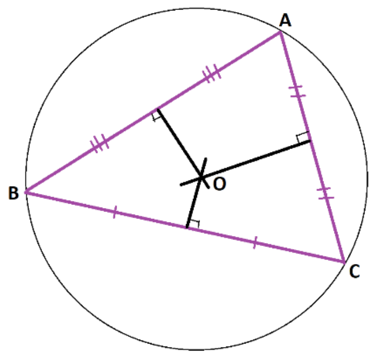
© James Tanton 2018

$$G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C .$$

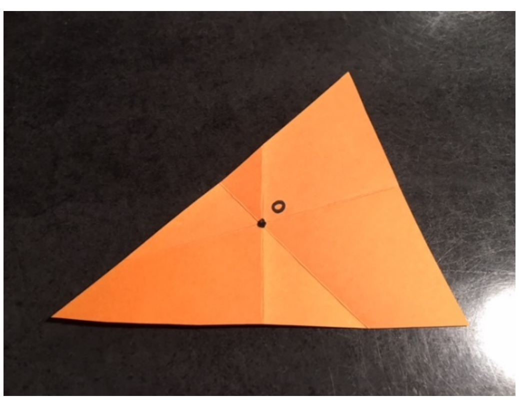
The centroid of the triangle in the opening puzzle has coordinates $G = (4,2)$.

THE CIRCUMCENTER

Each perpendicular bisector of a side of a triangle gives the set of points equidistant from two vertices of the triangle. Thus the point O at which just two perpendicular bisectors meet must be equidistant from all three vertices of the triangle and so lie on the third perpendicular bisector as well.



One can readily fold the perpendicular bisectors of a paper triangle and see the circumcenter appear inside the triangle if the triangle is acute.



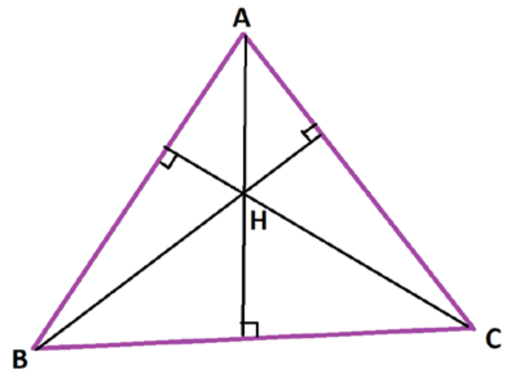
Because O is equidistant from all three vertices , it is possible to draw a circle with center O that circumscribes the triangle.

For this reason, O is call the *circumcenter* of the triangle.

I do not know a general formula for the circumcenter of a triangle in terms of its three vertices! (Do you?)

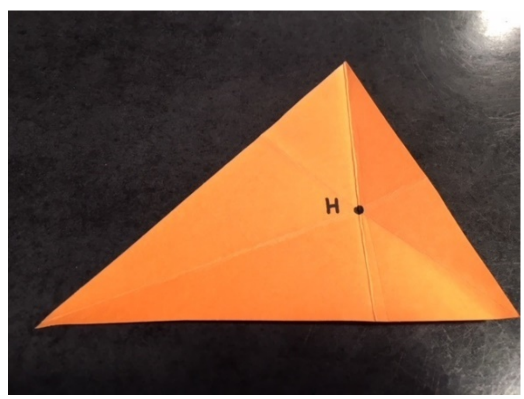
THE ORTHOCENTER

The three altitudes of a triangle meet at a common point O called the *orthocenter* of the triangle.



One can see that they are indeed concurrent by drawing lines through each vertex of the triangle parallel to its opposite sides to form a larger triangle. The altitudes of the original triangle are medians of the larger triangle and so must meet at a common point.

With a little care one can also fold the altitudes of a paper triangle.

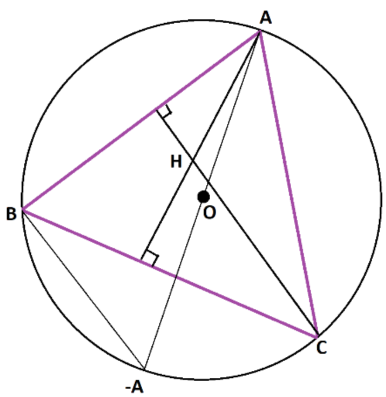


I know a formula for the orthocenter of a triangle only if we assume that the triangle

© James Tanton 2018

is placed in the plane so that its circumcenter O lies at the origin.

Deriving it is a bit sneaky.



Draw the circumscribing circle of the triangle with center O, the origin, and draw the diameter of this circle from A through O. Since O is the origin, this diameter meets the circle again at the point given by $-A$.

The triangle connecting A, B, and $-A$ is a right triangle, and so the shift from C to H is parallel to the shift from $-A$ to B.

To "reach" H from C we need to move by some (currently unknown) multiple of this shift. Thus we have

$$H = C + \lambda\left(B - (-A)\right)$$

for some value λ. That is,

$$H = \lambda A + \lambda B + C \text{ for some } \lambda.$$

By shifting from A or B instead we can similarly argue that

$$H = \mu A + B + \mu C \text{ for some } \mu$$

and

$$H = A + \nu B + \nu C \text{ for some } \nu.$$

To have three consistent equations we might guess we must have $\lambda = \mu = \nu = 1$ giving $H = A + B + C$. Could this be right?

Writing

$$A + B + C = C + 1\cdot\left(B - (-A)\right)$$

shows that $A + B + C$ is a point on a line from C perpendicular to $\overline{AB}$. And writing

$$A + B + C = B + 1\cdot\left(A - (-C)\right)$$

shows this point is also on a line from B perpendicular to $\overline{AC}$, and writing

$$A + B + C = A + 1\cdot\left(C - (-B)\right)$$

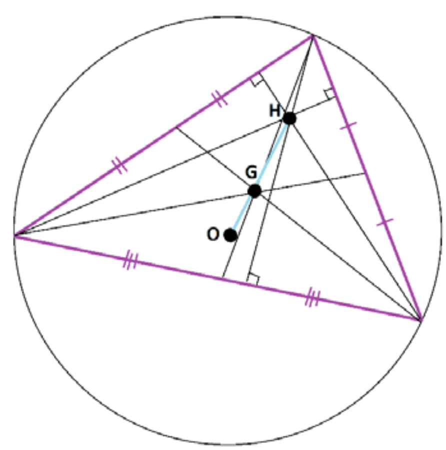
shows it is also on a line from A perpendicular to $\overline{BC}$.

That is, $A + B + C$ is a point on all three altitudes and so must indeed be the orthocenter H. We do have

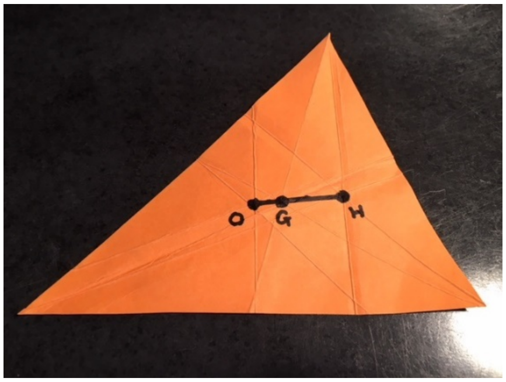
$$H = A + B + C.$$

The Euler Line

As a bonus we see that if O is the origin, then the formulas $H = A + B + C$ and $G = \frac{1}{3}(A + B + C)$ show that the points O, G, and H are collinear with G one-third of the way along $\overline{OH}$.



One can see this with paper-folding too.



RESEARCH CORNER

Find a general formula for the circumcenter O of a triangle.

Find a general formula for the orthocenter H of a triangle without the assumption that the circumcenter O lies at the origin.

(What are the coordinates of O and H for the opening puzzler?)

What is the formula for the incenter of a tetrahedron with vertices A, B, C, and D?

© 2018 James Tanton
tanton.math@gmail.com