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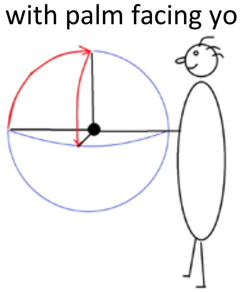
WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JUNE 2018

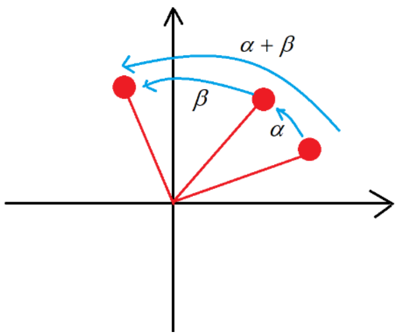
THIS MONTH'S PUZZLER:
Stretch out your right arm, palm upwards. Then bend at your elbow 90 degrees so that your forearm is vertical and your palm faces your face. Keeping your elbow fixed in place, turn your forearm 90 degrees counterclockwise. Your forearm is now horizontal across your chest with palm facing your chest.



If we regard the location of your elbow as the origin in space, your hand has just undergone two rotations about the origin in three-dimensional space. We like to believe that the composition two rotations is a rotation. So then, what is the single rotation you could have performed that takes your hand from the starting position of this exercise to its final position? In which direction does the axis of that rotation point? How many degrees of rotation are needed about that axis? (A single rotation of 90 degrees about the vertical axis, alas, is not the answer as this results in your palm remaining face up, not facing your chest.)

IS THE COMPOSITION OF TWO ROTATIONS ABOUT THE ORIGIN A ROTATION?

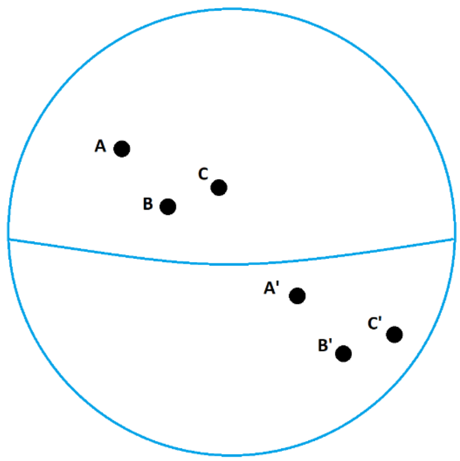
In two-dimensional space, the answer to this question is readily seen to be YES! A rotation of α degrees about the origin, followed by a rotation of β degrees about the origin matches a rotation of $\alpha+\beta$ degrees about the origin.



But the opening exercise shows that the answer to this question is far less obvious in three-dimensional space.

> Those who have studied quite advanced linear algebra might argue:
>
> *Every rotation about the origin in 3-space is represented by a matrix in* $SO(3)$*. This set of matrices is closed under multiplication, and so the composition of rotations is indeed a rotation. Done!*
>
> But I say in response: Hang on! Is it obvious that every element of $SO(3)$ is a rotation? Prove it! When you try you'll find yourself heading down a rabbit hole. It is true that every element of $SO(3)$ has a one-dimensional eigenvector space with eigenvalue 1 (this must be the axis of rotation), but the same result is not true for $SO(4)$. Something deeply subtle is afoot!

A rotation is a rigid motion: it preserves the distance between points. And every rigid motion in three-dimensional space (keeping the origin O fixed) is completely determined by what it does to three non-"collinear" points on a sphere about the origin. For instance, suppose A, B, and C are three such points and a rigid transformation r takes them to points A', B', and C' on the sphere. (They do lie on the sphere since the distances to the origin are preserved. Also, the spherical triangles ABC and $A'B'C'$ must be congruent triangles.)



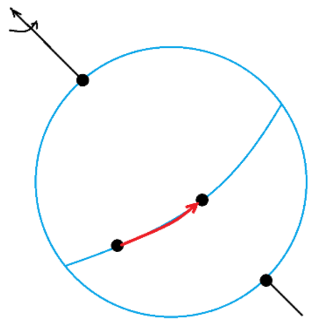
From these three image points we can now deduce where r takes any other given point P. Consider the six-faced polyhedron with vertices O, A, B, C, and P (even a degenerate one is okay). The polyhedron with vertices O, A', B', C', and $r(P)$ has exactly the same edge lengths and diagonal lengths (r preserves distances) and so the location $r(P)$ is completely determined with no ambiguity.

Upshot: *If you know what a rigid motion that fixes the origin does to a spherical triangle, then you, in principle, know the entire rigid motion.*

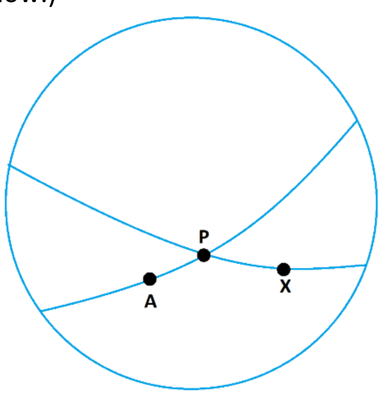
In other words, there is at most only one rigid motion fixing the origin that takes a given spherical triangle to another given congruent spherical triangle.

We can now prove that the composition of two rotations about the origin in 3-space is indeed another rotation.

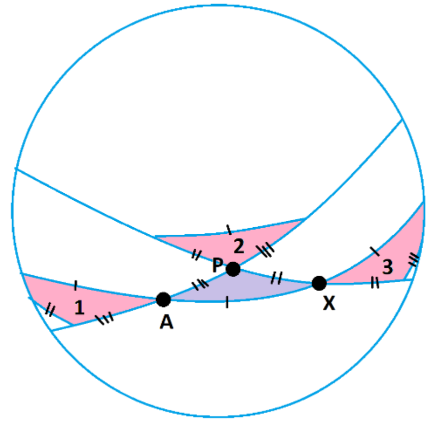
Consider a sphere about the origin. Any rotation fixes two antipodal points on the sphere (the ones through which the axis of rotation intersects the sphere) and has a great circle (an "equator") that is unchanged by the rotation, but points on this equator slide long it by some fixed amount given by the angle of the rotation.



Consider two rotations, the first with matching equator going through A and P shown in the diagram below, and the second with matching equator through P and X shown. I've also taken A and P and P and X to be distances apart along their respective equators matching **half** the angles of their respective rotations. (The choice to do this, I admit, is a bit mysterious right now.)



Now draw the equator through A and X, and construct the congruent spherical triangles shown.



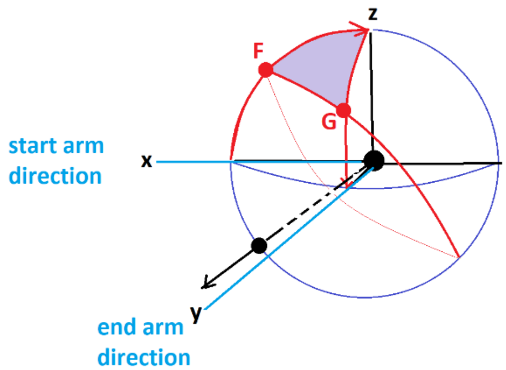
Our first rotation takes triangle 1 to triangle 2. (Do you see now why we initially went with half lengths?) Our second rotation takes triangle 2 to triangle 3. The composition of the two rotations is thus a rigid motion that takes triangle 1 to triangle 3. But the rotation with the equator through A and X at double the angle represented by the length between A and X also takes triangle 1 to 3. As there is at most one rigid motion that can move a spherical triangle this way, the composition the two rotations must match this single rotation!

Comment: I first learnt of this approach to analyzing 3-space rotations from Bernhard Elsner's (Math O'Man's) post: http://www.mathoman.com/en/index.php/1537-axis-and-angle-of-the-composition-of-two-rotations. Elsner points out there that the angle of rotation of a composite rotation is never more than the sum of the angles of the two original rotations. To see this, look at our purple triangle. The length of equator between A and X is less than the sum of the remaining two lengths of the spherical triangle.

THE ANSWER TO THE OPENING PUZZLER

Warning: Vector calculus!

My ability to sketch diagrams is severely lacking. But here is my attempt to illustrate our analysis as applied to the two 90-degree rotations of the opening puzzle.



Here the appropriate purple triangle has vertices F and G at the midpoints of two quarter equators. If we label x-, y-, and z-axes as shown and assume the sphere has radius 1 unit, then coordinates of these points are $F=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$ and $G=\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. The angle θ between the vectors $\overrightarrow{OF}$ and $\overrightarrow{OG}$ satisfies

$$\cos\theta=\frac{\overrightarrow{OF}\cdot\overrightarrow{OG}}{1\cdot 1}=\frac{1/2}{1}=\frac{1}{2}$$

giving $\theta=60^\circ$ and so the composite rotation is through an angle of 120°.

The coordinates of the origin, F, and G each satisfy the equation $x+y-z=0$ and so this is the equation of the plane that contains the equator through F and G. This plane has normal vector $<1,1,-1>$, which shows the axis of rotation for the composite rotation lies some angle below the diagonal line between the positive x- and y-axes. (One can check that it is down an angle of $\cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{3}}\right)\approx 35^\circ$.)

If you are following what I am trying to describe here you can try acting out a 120-degree rotation about an axis at some angle down from the ray between where your forearm starts and where you want it to end and see that this analysis is plausible.

> **Exercise:** For something easier:
>
> *Starting with outstretched right arm, palm up, perform the two rotations of the opening puzzle but in reverse order. Do you see that you end up with your forearm vertical, palm facing left?*
>
> This different result shows that two rotations about the origin in 3-space need not commute. (Do rotations about the origin in 2-space commute?)

RESEARCH CORNER

1. Was vector calculus actually needed in the previous section? Is there an argument of symmetry that allows you to deduce that arc $\overset{\frown}{FG}$ is one third of a semi-equator and thus the angle of rotation of the composite rotation is 120°? (We can certainly see that the axis of the composite rotation lies at some angle below the diagonal ray between the positive x- and y-axes.)

2. Is there a natural way to see that for any two congruent spherical triangles on a sphere there must be a rotation about the center of the sphere that maps one onto the other? (Or must one establish this through a composition of rotations?)

(Does every matrix in $SO(3)$ indeed represent a rotation about the origin? And is each and every rotation so represented?)

3. Stretch out your right arm again, palm up. Keeping your elbow fixed in place, perform the same two rotations of the opening puzzle: rotate your forearm 90 degrees so that your forearm is vertical, palm facing your face, and then rotate your forearm 90 degrees so that it is horizontal across your chest, palm facing your chest. Now perform one more rotation, a 90-degree turn about a vertical access so that your forearm returns to its starting position, but now with the palm of your hand facing to the left. Actually perform each of these three 90-degree rotations again. You'll see that you end up back in your starting position but with your palm facing downwards.

Can you explain why the composition of these six 90-degree rotations is just a 180-degree rotation about the axis of your original outstretched arm?

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