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WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

June 2017

THIS MONTH'S PUZZLERS:
Here are two fun coin-tossing puzzles. Pause and mull on these before reading on: this essay launches right into their solutions.

COIN TOSSING I
Each day for a (365-day) year I shall toss a fair coin until I first see a head appear. On some days this will be on the first toss, on other days I will see tails one or more times before first seeing a head.

a) At the end of the year I know I will see a total of 365 heads. (One a day.) Do I expect to see more tails than this? Less?

b) Each day I shall record the percentage of heads I see among my tosses. This percentage could be 100% (getting a head on the first toss) or 50% (getting a tail and then a head), or less. At the end of the year I shall compute the average value of these percentages. What value do I expect that average to be?

c) How many tosses in total do I expect to make during the year?

COIN TOSSING II
In another 365-day year I shall toss a fair coin 10 times in a row each and every day.

And each day I shall count the number of pairs of consecutive heads I see, and the number of heads/tails pairs I see.

For example, for the string

H H T T H T T T H T

I would report a count of one HH pair and three HT pairs.

d) Over the course of the year do I expect to see more HH pairs than HT pairs, or less?

e) Suppose, instead, I count the number of HHH and TTH triples I see each day. For example, the string above there are no HHH triples and two TTH triples.

Over the course of the year, which total count do I expect to be largest?

COIN TOSSING I

Suppose I toss a coin until a head appears each day for 365 days, recording the results in a big list.

T T H
H
T T T T H
T T H
H
H
T H
T H
H
⋮

We can argue that, with a fair coin, about half of these lines will be one entry long: we toss a head right away and count 0 tails; and about one-quarter of these lines will be two entries long, TH, to contain 1 tail; and one-eighth will be three entries long, TTH, to contain 2 tails; and so on.

And we could write an infinite sum to compute the expected number of tails we should see. That sum is

$$0\times\left(\frac{1}{2}\times 365\right)+1\times\left(\frac{1}{4}\times 365\right)+2\times\left(\frac{1}{8}\times 365\right)+3\times\left(\frac{1}{16}\times 365\right)+\cdots$$
$$=365\times\left(0\times\frac{1}{2}+1\times\frac{1}{4}+2\times\frac{1}{8}+3\times\frac{1}{16}+\cdots\right).$$

And we could compute the sum

$$0\times\frac{1}{2}+1\times\frac{1}{4}+2\times\frac{1}{8}+3\times\frac{1}{16}+\cdots$$

by teasing it apart

$$\begin{aligned}&\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\\&\quad+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\\&\qquad+\frac{1}{16}+\frac{1}{32}+\cdots\\&\qquad\quad+\frac{1}{32}+\cdots\\&\qquad\qquad\vdots\end{aligned}$$

to see it equals

$$\begin{aligned}&\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)\\&\quad+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)\\&\qquad+\frac{1}{8}\left(\frac{1}{2}+\frac{1}{4}+\cdots\right)\\&\qquad\quad+\cdots\\&=\frac{1}{2}\times 1+\frac{1}{4}\times 1+\frac{1}{8}\times 1+\cdots=1.\end{aligned}$$

We would thus conclude that, on average, we'd expect to see $365\times 1=365$ tails appearing among the 365 lines of data.

The expected total number of tails is equal to the guaranteed total number of heads.

But this approach to this conclusion was too hard! Argue instead this way:

Among any collection of specified coin tosses we expect, on average, an equal number of heads and tails among them.

Consequently …

For all the first tosses that appear in the data list, we'd expect an equal number of Hs and Ts among them. For all the second tosses that appear in the data, we'd expect an equal number of Hs and Ts. The same among all the third tosses in the data, and in among the fourth tosses in the data, and so on. This accounts for all the data. Thus we expect an equal number of Hs and Ts in the data list.

As 365 heads is certain, we expect that many tails as well.

With 365 heads and an expected count of 365 tails, this means we expect to make a total of 730 tosses in all over the year. This answers part c) of the puzzle.

I personally don't know how to answer part b), alas, with conceptual ease: I must resort to an infinite sum and, further, use the tools of calculus to evaluate it. Here's how.

Warning: Calculus!

Half the time we expect the fraction 1 of our tosses to be heads, and a quarter of the time the fraction $\frac{1}{2}$ of our tosses to be heads, and one-eighth of the time the fraction $\frac{1}{3}$ of our tosses to be heads, and so on. Thus the average fraction of heads we expect is

$$\frac{1}{2}\times 1+\frac{1}{4}\times\frac{1}{2}+\frac{1}{8}\times\frac{1}{3}+\frac{1}{16}\times\frac{1}{4}+\cdots.$$

This is the infinite sum

$$f(x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$

with $x=1/2$.

Taking the derivative of this series feels compelling as it will clear away the denominators.

$$f'(x)=1+x+x^2+x^3+\cdots ,$$

which I recognize as the geometric series:

$$f'(x)=\frac{1}{1-x} .$$

This means that we have

$$f(x)=-\ln(|1-x|)+C .$$

Since $f(0)=0$, we must have $C=0$.

So

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots=-\ln(|1-x).$$

Putting in $x=1/2$ gives the average value of the fraction of heads we expect. It is

$$-\ln\left(\frac{1}{2}\right)=\ln 2\approx 0.69 .$$

Comment: We should worry about the radii of convergence of these series, worry about whether or not we are allowed to differentiate and integrate these series, and whether or not $x=1/2$ is a permissible input for these series. Please do feel free to resolve these worries.

So we are left with a disturbing conclusion: At the end of the year we expect an equal number of heads and tails, and so expect, in total, a 50% fraction of heads. But if we go with the day-by-day reports of the fraction of heads that appear, we expect an average daily report of 69% heads.

Is this a contradiction?

COIN TOSSING 2

Suppose now, for each day of the year, I toss a coin ten times in a row and list all the ten-string results.

HTTHHTTHTT
HHHTHTTTTT
TTHHTHHTTH
⋮

Of all the strings that have H in the first position, on average, half will have an H in the second position as well and half will have a T there.

Of all the strings that have H in the second position, on average, half will have an H in the third position and half will have a T there.

Of all these strings that have H in the third position, on average, half will have an H in the fourth position and half will have a T there.

And so on.

Thus, on average, we expect to see an equal total count of HH pairs as HT pairs, answering puzzle d).

Suppose I now count the number of HHH triples and TTH triples that appear in each string.

Of all these strings that have H in the tenth position, on average, a quarter of them will have HH in the two positons before it, and a quarter will have TT instead.

Of all these strings that have H in the ninth position, on average, a quarter of them will have HH in the two positons before it, and a quarter will have TT instead.

And so on.

Thus, to answer e), on average we expect to see equal counts of HHH triples and TTH triples.

RESEARCH CORNER

1. How do the results of these puzzles change if we work with a biased coin?

2. Is it possible to compute the answer to part b) without resorting to the tools of calculus?

3. Fix values N and k with $k \leq N$. Suppose we toss a coin N times in a row each day for a year. Do we expect all possible strings of k Hs and Ts to appear the same number of times over the course of the year?

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