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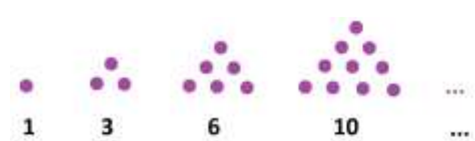
WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

August 2017

THIS MONTH'S PUZZLER:

The sequence of *triangular numbers* begins, 1, 3, 6, 10, 15, …. These are the numbers that arise as counts of pebbles that appear in triangular arrays.



Let $T(n)$ denote the n th triangular number. Thus $T(1)=1$, $T(2)=3$, and so on.

Show the triangular numbers satisfy these two lovely relations

$$T(a+b)=T(a)+T(b)+ab$$

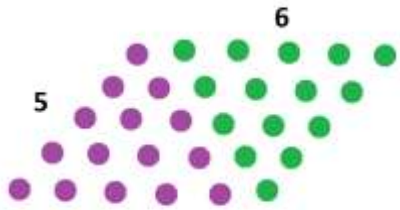
$$T(ab)=T(a)T(b)+T(a-1)T(b-1)$$

for integers $a,b\geq 1$ (with the understanding that $T(0)=0$).

Prove these relations just by drawing pictures.

THE LOVELY RELATIONS

Placing together two copies of the same triangular array gives a (tilted) oblong.



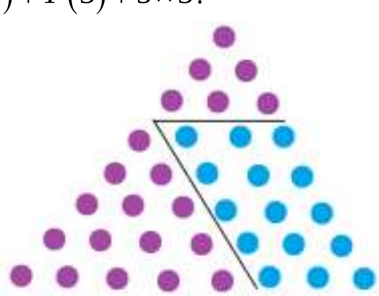
Seeing $T(5)$ *as half of* 5×6.

We see that $T(n)=\frac{n(n+1)}{2}$.

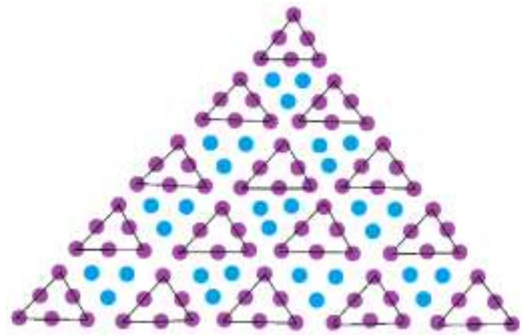
We can certainly use algebra to establish the relations
$T(a+b)=T(a)+T(b)+ab$
and
$T(ab)=T(a)T(b)+T(a-1)T(b-1)$.

Alternatively, let's let pictures speak their thousand words.

This picture shows that $T(5+3)$ equals $T(5)+T(3)+5\times 3$.



The next picture shows a diagram for $T(15)$ as composed of a triangular number of purple triangles (of type $T(3)$) and a triangular number of blue triangles (of type $T(2)$).



There are $T(5)$ purple triangles and $T(4)$ blue triangles. Thus $T(15)$ equals

$$T(5)\times T(3)+T(4)\times T(2).$$

Both pictures speak to "higher truths," those truths being that
$T(a+b)=T(a)+T(b)+ab$
and
$T(ab)=T(a)T(b)+T(a-1)T(b-1)$ for all integers $a,b\geq 1$.

Question: The square numbers, $S(n)=n^2$, satisfy
$S(a+b)=S(a)+S(b)+2ab$.
(Establish this with a picture!)

Find a sequence of numbers, $P(n)$, with $P(0)=0$, $P(1)=1$ satisfying
$P(a+b)=P(a)+P(b)+3ab$.

Find a sequence of numbers, $H(n)$, with $H(0)=0$, $H(1)=1$ satisfying
$H(a+b)=H(a)+H(b)+4ab$.

Keep going! (What geometric numbers are you discovering?)

Further : Find a sequence of numbers, $A(n)$, with $A(0)=0$, $A(1)=1$ satisfying

$$A(a+b)=A(a)+A(b)+1\frac{1}{2}ab.$$

Are your numbers halfway between being square and triangular?

Further Still: Find a sequence of numbers, $K(n)$, with $K(0)=0$, $K(1)=1$ satisfying

$$K(a+b)=K(a)+K(b)-ab.$$

UNIQUENESS?

Is the sequence of triangular numbers $T(n)$ the only integer sequence satisfying $T(ab)=T(a)T(b)+T(a-1)T(b-1)$?

Well, the sequence that is a constant value of zero satisfies this relation, so we need a non-trivial starting condition. Let's require that $T(1)=1$.

Since the relation has two variables, we likely need two starting values. But notice that from $T(1)=1$ and

$$T(1)=T(1\times 1)=T(1)^2+T(0)^2$$

we get $T(0)=0$ automatically. To get us off the ground, we need a different, second starting condition. Let's require then that $T(2)=3$.

[There's a logical gap here: Must we set $T(2)$ equal to three?]

So here is our question (for now, at least).

If a sequence of integers $T(n)$ satisfies $T(ab)=T(a)T(b)+T(a-1)T(b-1)$ for all integers $a,b\ge 1$ with $T(1)=1$ and $T(2)=3$, must that sequence be the sequence of triangular numbers?

Let's play with the relation to determine more required values of the sequence. So far we have

$$T(0)=0$$
$$T(1)=1$$
$$T(2)=3.$$

We have

$$T(4)=T(2)^2+T(1)^2=9+1=10.$$

We've skipped over the value of $T(3)$. Let's set $T(3)=d$ for now.

It then follows that

$$T(9)=T(3)^2+T(2)^2=d^2+9,$$
$$T(6)=T(2)T(3)+T(1)T(2)=3d+3,$$

and

$$T(8)=T(2)T(4)+T(1)T(3)=30+d.$$

We don't have $T(5)$ and $T(7)$.

$T(0)=0$	$T(5)=?$
$T(1)=1$	$T(6)=3d+3$
$T(2)=3$	$T(7)=?$
$T(3)=d$	$T(8)=30+d$
$T(4)=10$	$T(9)=d^2+9$

Actually computing $T(12)$ two ways gives some help.

$$T(12)=T(2)T(6)+T(1)T(5)=9d+9+T(5)$$
$$=T(3)T(4)+T(2)T(3)=13d$$

So

$$T(5)=4d-9.$$

Your Turn: Compute $T(7)$ in terms of d by looking at $T(16)$ two ways. Compute $T(10)$ too.

It looks like we can build up the whole sequence of integers in terms of d.

Suppose we have formulas or numbers for all the terms $T(1)$, $T(2)$, …, $T(p-1)$ for $p>3$. We then can compute the next term $T(p)$ as well.

Here's how.

If $p=a\times b$ is composite, then use $T(p)=T(a)T(b)+T(a-1)T(b-1)$. All the necessary early values of the sequence are known.

If p is an odd prime, then we can compute $T(p+1)$ as $p+1$ is composite and composed of factors smaller than p. (We can use $p+1=2\times\frac{p+1}{2}$.) Now compute $T(2(p+1))$ two ways. One way gives

$$T(2)T(p+1)+T(1)T(p)$$
$$=3T(p+1)+T(p).$$

Another way gives

$$T(2)T\left(\frac{p+1}{2}\right)+T(1)T\left(\frac{p-1}{2}\right)$$
$$=3T\left(\frac{p+1}{2}\right)+T\left(\frac{p-1}{2}\right).$$

Then $T(p)$ can be computed as

$$3T\left(\frac{p+1}{2}\right)+T\left(\frac{p-1}{2}\right)-3T(p+1).$$

(This is how I computed $T(5)$: by looking at $T(12)$ two ways.)

So this means all the values of the sequence are fixed and depend only on the value of d.

But what is the value of d?

Let's compute $T(18)$ two ways.

$$T(18)=T(2)T(9)+T(1)T(8)$$
$$=3d^2+27+30+d$$
$$T(18)=T(3)T(6)+T(2)T(5)$$
$$=3d^2+3d+12d-27$$

Thus it follows that $14d=84$ and $d=6$.

Thus there is only one possible value of d giving a sequence satisfying the desired initial conditions and general relation. And we know one sequence that works. Thus the sequence of triangular numbers is indeed the only sequence of integers that satisfies the required conditions.

Question: Are the triangular numbers the only sequence of numbers satisfying $T(a+b)=T(a)+T(b)+ab$ with $T(1)=1$?

RESEARCH CORNER

A sequence of real numbers satisfies $T(ab)=T(a)T(b)+T(a-1)T(b-1)$ with $T(1)=1$.

If we set $T(2)=c$ and $T(3)=d$, then we can show that each term of the sequence is determined by c and d.

Are $c=3$ and $d=6$ the only two integers that ensure that each term of the sequence is an integer? (Examining $T(18)$ is again enlightening.)

tanton.math@gmail.com