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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

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THIS MONTH'S PUZZLER:
I am personally charmed by this not-so-little question:

How many right triangles with integer side lengths have one edge of length 2^{100} units?

A PRECURSOR QUESTION

I also very much like this question!

Two non-zero square numbers sum to a power of two. What can you say about those two square numbers?

It turns out that the two square numbers must be identical and each a power of two.

To see why, suppose that $m^2 + n^2 = 2^r$ for some non-zero integers m and n and non-negative integer r.

We first note that r cannot be zero as there are no solutions to $m^2 + n^2 = 1$ with m and n each non-zero. If $r = 1$, then $m^2 + n^2 = 2$ has just one solution, namely, $m = n = 1$, with m and n (and hence m^2 and n^2) identical powers of two.

Let's assume then that $r \geq 2$. In this case, 2^r is a multiple of four.

Every even square number is also a multiple of four ($(2k)^2 = 4k^2$) and every odd square number is one more than a multiple of four ($(2k+1)^2 = 4(k^2+k)+1$).

Consequently, since $m^2 + n^2 = 2^r$ is a multiple of four we must have that m and n are each even.

Write $m = 2m_1$ and $n = 2n_1$. Then $4m_1^{\;2} + 4n_1^{\;2} = 2^r$ gives $m_1^{\;2} + n_1^{\;2} = 2^{r-2}$, which is a sum of two squares equaling a power of two again.

As before, there are no solutions if $r-2=0$, so it must be that $r-2 \geq 1$.

If $r-2=1$, then $m_1^{\;2} + n_1^{\;2} = 2$ forces $m_1 = n_1 = 1$ and hence $m = n = 2$ and we have two identical powers of two. (And it follows that m^2 and n^2 are identical powers of 2 too.)

If $r-2 \geq 2$, then $m_1^{\;2} + n_1^{\;2} = 2^{r-2}$ forces us to conclude that m_1 and n_1 are each even.

Write $m_1 = 2m_2$ and $n_1 = 2n_2$ and observe that we then have $m_2^{\;2} + n_2^{\;2} = 2^{r-4}$, a sum of two squares equaling a power of two again.

There are no solutions if $r-4=0$, so it must be that $r-4 \geq 1$.

If $r-4=1$, then $m_2^{\;2} + n_2^{\;2} = 2$ forces $m_2 = n_2 = 1$, and hence $m = n = 4$,and we have two identical powers of two. (And it follows that m^2 and n^2 are identical powers of two too.)

If $r-4 \geq 2$, then $m_2^{\;2} + n_2^{\;2} = 2^{r-2}$ forces us to conclude that m_2 and n_2 are each even.

Write $m_2 = 2m_3$ and $n_2 = 2n_3$ and observe that we then have $m_3^{\;2} + n_3^{\;2} = 2^{r-6}$.

And the cycle of reasoning continues.

This cycle cannot continue indefinitely (each equation we obtain works with a smaller power of two), so we conclude that there must be a number k such that $m = 2^k m_k$ and $n = 2^k n_k$ with $m_k^{\;2} + n_k^{\;2} = 2^{r-2k} = 1$. This forces $m_k = n_k = 1$ and so m and n are each 2^k, identical powers of two, and m^2 and n^2 are each 2^{2k}. This proves the claim.

Phew!

THE ANSWER TO THE MAIN QUESTION

Now let's attend to the opening puzzle.

Let a, b, and c be the side lengths of an integer right triangle with c the length of the hypotenuse.

The Pythagorean theorem tells as that $a^2 + b^2 = c^2$. The integers (a,b,c) thus form a *Pythagorean triple*. If these integers share no common factor other than 1, then they are called a *primitive Pythagorean triple.*

Any given Pythagorean triple is either primitive or a "multiple" of a primitive Pythagorean triple: just divide each entry by the largest common factor of the entries.

For example, $(30,40,50)$ is a Pythagorean triple and dividing each entry by 10 gives the primitive triple $(3,4,5)$.

Around 300 B.C.E. the great Greek geometer completely categorized primitive Pythagorean triples. He proved that if

(a,b,c) is a primitive triple, then there are integers $m > n$, one even and the other odd, so that

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

with the understanding that the first two entries of the triple might have to be reordered.

This celebrated result is a tad tricky to prove, but proofs of it can easily be found on the internet.

Notice here that a and c are necessarily odd integers and b is even. Let's assume that whenever we write primitive Pythagorean triple, the first entry we list is odd and the second entry is even, with the third entry, necessarily odd, the largest value of the three.

Let's first ask:

How many primitive Pythagorean triples (a,b,c) contain 2^{100} as an entry?

Since b is the only even entry, we must have $b = 2^{100}$.

Writing $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ for some integers $m > n$ with one even, the other odd, we see then that $mn = 2^{99}$. There is only one option then for the values of m and n. We must have $m = 2^{99}$ and $n = 1$ yielding the one primitive triple $\left(2^{198} - 1, 2^{100}, 2^{198} + 1\right)$.

In general:

For each $r \ge 2$ there is precisely one primitive Pythagorean triple with entry 2^r, namely, $\left(2^{2r-2} - 1,\ 2^r,\ 2^{2r-2} + 1\right)$.

Exercise: Explain why there is no primitive Pythagorean triple with 2 as an entry.

Explain why there is no primitive Pythagorean triple with 1 as an entry.

The one primitive Pythagorean triple with 4 as an entry is $(3,4,5)$.

Exercise: Explain why there is no non-primitive Pythagorean triple with 4 as an entry.

Doubling the terms of this triple yields the non-primitive triple $(6,8,10)$ with entry 8. Plus there is one primitive triple with entry 8, namely, $(15,8,17)$. These two are the only Pythagorean triples with 8 as an entry.

Doubling the terms of these triples gives two non-primitive Pythagorean triples with entry 16, namely, $(12,16,20)$ and $(30,16,34)$. Plus there is one primitive triple with entry 16, $(63,16,65)$. These three are the only Pythagorean triples with 16 as an entry.

In general:

For each $r \ge 2$ there are precisely $r-1$ Pythagorean triples with 2^r as an entry.

(Each unique primitive Pythagorean triple with entry 2^k for $k = 2,3,...,r$ can be scaled by a power of two to produce a triple with entry 2^r. Thus there are at least $r-1$ Pythagorean triples with entry 2^r. There can be no other triple for any new triple is a multiple of a primitive triple with a multiple that is a power of two, and that scaled copy of that primitive triple has already been accounted for.)

So this answers the opening puzzle.

There are precisely 99 integer right triangles with one side of length 2^{100}*. In each example, the side of length* 2^{100} *is a leg of the triangle.*

RESEARCH CORNER

How many integer right triangles have a side of length 3^{100} ?

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