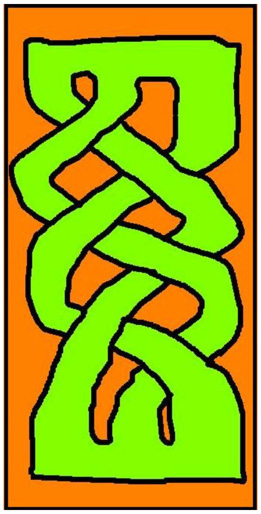

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WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

January 2018

THIS MONTH'S PUZZLER:

Take a string 4 units long composed of 0 s and 1s and repeat it five times to make a string 20 units long. (For example, the string 1001 gives 10011001100110011001.)
Also take a string of 0 s and 1s 5 five units long and repeat it four times to make a string 20 units long. (For example, 10100 gives 10100101001010010100.)

Place the two strings on top of one another.

1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1
1 0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0 1 0 0

What do you notice about the number of times the zeros align in the double string?

Try this with several examples until you can identify a general phenomenon.

Also try a triple string of 0 s and 1s 60 units long composed of a string of 3 symbols repeated 20 times, a string of 4 symbols repeated 15 times, and a string of 5 symbols repeated 12 times. What do notice about the count of times zeros align in all three strings?

THE CHINESE REMAINDER THEOREM

In a row of sixty houses along one side of a street, the door of the first house is painted yellow, the door of the second house is painted red, the door of the third house green, and this cyclic pattern of yellow-red-green continues along all sixty houses in the row. The window frames of the houses cycle in the colors yellow, red, green, blue along the row; and the roofs of the houses cycle in colors yellow, red, green, blue, purple.

Is there a house with a red door, yellow window frames, and blue roof?

To answer this question, first notice that there can be at most one house with these three characteristics. To see this, number the houses 1 through 60 along the road. Any two houses with same color door differ in house number by a multiple of three, any two houses with same color window frames differ in house number by a multiple of four, and any two houses with same color roofs differ in house number by a multiple of five. Thus any two houses with doors, windows, and roofs matching in color differ in house number by a multiple of three, four, and five, that is, by a multiple of 60. As there are only 60 houses, there are no two different houses with house numbers that differ by such a multiple. Thus there is at most one house with these three particular color characteristics.

Now we can argue that each and every possible combination of door, window frame, and roof colors must appear. To see this notice that there are $3 \times 4 \times 5 = 60$ possible color combinations. If any one of these combinations is missed, then there must be two houses with precisely the same colorings to account for the deficiency, which we just proved is impossible.

So yes, a house with any desired color combination of door, window frames, and roof does appear.

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The formal Chinese Remainder Theorem in mathematics is precisely this result, but usually phrased in terms of remainders upon division.

If n_1, n_2, …, n_k are numbers with no two sharing a common factor other than 1 and, for each i, we are given a number $0 \le a_i < n_i$, then there is a number x whose remainder upon division by n_i is precisely a_i for each i.

$$\begin{aligned} &x \div n_1 \text{ has remainder } a_1 \\ &x \div n_2 \text{ has remainder } a_2 \\ &\vdots \\ &x \div n_k \text{ has remainder } a_k \end{aligned}$$

Moreover, any two numbers x that satisfy this property differ by a multiple of $n_1 \times n_2 \times \cdots \times n_k$.

To prove this, just think of each number n_i as an attribute of some feature of a house—a door color, a verandah color, a chimney color—which can be one of a_i colors and imagine $n_1 \times n_2 \times \cdots \times n_k$ houses in a row.

Challenge: Find a number that leaves a remainder of 1 upon division by five, a remainder of 2 upon division by six, and a remainder of 2 upon division by seven.

YRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRGYRG
YRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGBYRGB
YRGBPYRGBPYRGBPYRGBPYRGBPYRGBPYRGBPYRGBPYRGBPYRGBPYRGBPYRGBP

THE OPENING PUZZLER

Take any string of four with two 0 s and any string of five with three 0 s. Then I claim in the double string twenty units they produce, the zeros will align precisely $2\times 3 = 6$ times. This is certainly the case for the example in the opening puzzler and these two examples.

0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
0 0 0 1 1 0 0 0 1 1 0 0 0 1 1 0 0 0 1 1

1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0
1 0 0 0 1 1 0 0 0 1 1 0 0 0 1 1 0 0 0 1

By the Chinese Remainder Theorem each and every element of the four-term sequence will align with each and every element of the five-term sequence exactly once.

A1 A2 A3 A4 A1 A2 A3 A4 A1 A2 A3 A4 A1 A2 A3 A4 A1 A2 A3 A4
B1 B2 B3 B4 B5 B1 B2 B3 B4 B5 B1 B2 B3 B4 B5 B1 B2 B3 B4 B5

Thus its first zero will align with the first, second, and third zeros of the five term-sequence each once, as will its second zero. The zeros will thus align $2\times 3 = 6$ times.

In general, if there are a_1 zeros among a repeating block of n_1 terms and a_2 zeros among a repeating block of n_2 terms, then zeros will align $a_1 a_2$ times in the double string $n_1 n_2$ units long (provided n_1 and n_2 share no common factor other that 1).

The analogous result holds for three cycles making a triple string, and so on.

Challenge: Our result relies on the premise that the two cycle lengths n_1 and n_2 have the property that any number that is a multiple of both n_1 and n_2 must actually be a multiple of their product $n_1 n_2$. This is not the case for $n_1 = 4$ and $n_2 = 6$, for instance, or for any pair of numbers sharing a common factor larger than 1.

Give an example of a four-cycle and a six-cycle where the expected number of aligned zeros fails to appear in the double string they yield.

Research: Is there a general theory to be developed about the number of expected alignment of zeros for two cycle lengths sharing a non-trivial common factor?

RESEARCH CORNER

1. There $\binom{4}{a} = \frac{4!}{a!(4-a)}$ four-term sequences of 0 s and 1s with a zeros, and $\binom{5}{b} = \frac{5!}{b!(5-b)!}$ five-term sequences with b zeros. Each choice of sequences gives a pattern of ab aligned zeros among the twenty terms of the double string they yield. There are a total of $\binom{20}{ab} = \frac{20!}{(ab)!(20-ab)!}$ possible patterns for these locations.

Does every possible pattern of zero-alignments appear? Can one classify the patterns we shall see among all possible four- and five-term sequences?

2. We can predict how many times 0 s will align and, by the same reasoning, how many times 1s will align. Can we predict the count of times consecutive pairs of symbols will align? (There are 5 such pairs in the example shown.) Can we predict the length of the longest repeated substring? (It is 1001 in the example shown.)

1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1
1 0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0 1 0 0

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