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WOWZA! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

February 2018

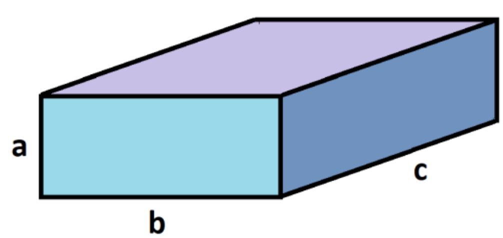
THIS MONTH'S PUZZLER:

A rectangular box has six rectangular faces each of the same area. Must that box be a cube?

A tetrahedral box has four triangular faces each of the same area. Must that box be a regular tetrahedron (that is, have four congruent equilateral triangle faces)?

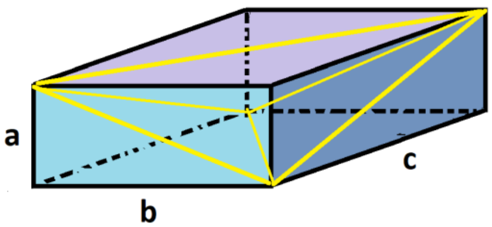
POLYHEDRON SYMMETRY

Consider a rectangular box with side lengths a, b, and c.



If each face has the same area, then we have $ab = bc = ac$. The first equality gives $a = c$ and the second $a = b$. Thus all three side-lengths are the same and the figure is indeed a cube.

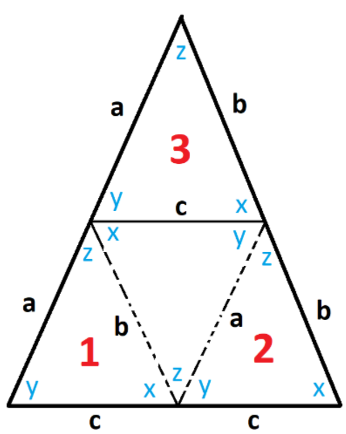
One might suspect that a tetrahedron with faces of the same area must be regular too, but this is not actually the case! Look at the tetrahedron formed by the diagonals of a non-cube box.



Each triangular face as side lengths $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$, and $\sqrt{a^2+c^2}$, and thus all faces are congruent and so have the same area, but the faces are not equilateral if a, b, and c are not all equal.

> **Challenge:** Find another tetrahedron with congruent faces situated in a rectangular box by connecting the midpoints of some edges of the box.

One can obtain other examples of such tetrahedra by cutting out a paper triangle with three acute angles. Connecting midpoints of its sides with line segments divides the triangle into four congruent sub-triangles.

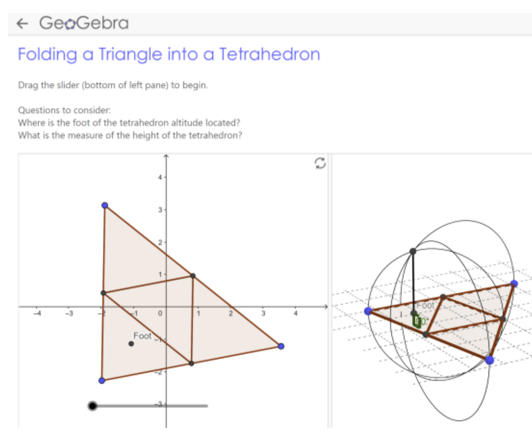


Lay the paper triangle on a table top and fold the paper along the two dotted lines to tilt sub-triangles 1 and 2 into three-dimensional space. Since angle z is smaller than 90°, we have that $x+y>90^\circ$, and so these two triangles can't fold back flat onto the table top without overlap. So there is an intermediate position in three-dimensional space where these two triangles meet along their edges of common length c. Now fold along the third midpoint line to tilt triangle 1 into three-dimensional space. It's side-lengths a and b align perfectly with the space formed by triangles 1 and 2 to allow us make a tetrahedron.

Thus from any acute triangle we can construct a non-regular tetrahedron with four congruent faces necessarily of the same area.

> **Challenge:** A previous diagram shows how to find a tetrahedron with congruent faces sitting inside any given rectangular box. Are the faces of such tetrahedra necessarily acute triangles?
>
> Can every tetrahedron with congruent acute triangular faces be situated in a rectangular box in this way?

Steve Phelps has created a super interactive GeoGebra app of this folding construction for you to play with. https://www.geogebra.org/m/CMezbUys



(What happens if you play with an obtuse triangle?)

RESEARCH CORNER

Is there an example of a tetrahedron with four faces of the same area with at least one face an obtuse triangle?

Find a general formula for the volume of tetrahedron with four congruent acute-triangle faces.

If a rectangular box has six faces of the same perimeter, it must be a cube. (Why?) What can you say about a tetrahedron with four triangular faces of the same perimeter?

The Wikipedia page on disphenoids, https://en.wikipedia.org/wiki/Disphenoid, makes a number of interesting claims about the tetrahedra discussed in this essay. Are the mathematical claims true?

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