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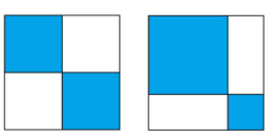
WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

December 2017

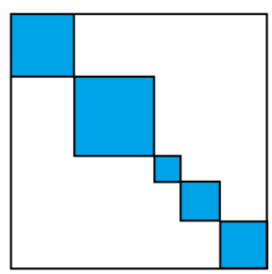
THIS MONTH'S PUZZLER:

Puzzle 1: Two squares of equal size arranged on the diagonal of an enveloping square take up half the area of the enclosing square.



Prove that squares of unequal size sitting on the diagonal of an enveloping square are sure to take up more than half its area.

Are five squares not all identical in size sitting on the diagonal of an enveloping square sure to take up more than one fifth of the area of that square?



Puzzle 2: The price of gloop varies from day to day, but is usually somewhere around $1 per gallon. I am going to buy about a gallon of gloop each day for a month. I am not really interested in the

exact total amount of gloop I purchase (it will be around 30 gallons) nor the total amount of money I spend (it will be about $30), but instead the best overall deal I get for the total amount of gloop received for the total amount of money spent.

Will I get a better deal, overall, by buying one gallon of gloop each day for a month no matter its price, or by buying one dollar's worth of gloop each and every day?

SOLVING PUZZLE 1

We can solve the first part of the first puzzler with algebra. It is asking to show that if a and b are two distinct positive real numbers, then we have

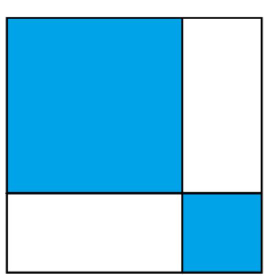
$$a^2 + b^2 > \frac{(a+b)^2}{2}.$$

This is algebraically equivalent to showing that $2a^2 + 2b^2 > a^2 + 2ab + b^2$, that is, that

$$a^2 - 2ab + b^2 > 0.$$

And this is equivalent to the claim that $(a-b)^2 > 0$, which is patently true if a and b are distinct values.

Challenge: Show directly on this picture that the area in blue simply must add to more than half the area of the large square.



We can use very similar algebra to generalize this result.

Lemma: *For positive real numbers a, b, x, and y we have*

$$\frac{a^2}{x} + \frac{b^2}{y} \ge \frac{(a+b)^2}{x+y},$$

with equality only if $\frac{a}{x} = \frac{b}{y}$.

Swift Proof: The given equality is equivalent to the statement $(ay - bx)^2 \ge 0$.

Observe that setting $x = 1$ and $y = 1$ gives our previous result.

This generalization is lovely as we can now see how to extend the result even further. For example, for positive real numbers we have

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \ge \frac{(a+b+c)^2}{x+y+z}$$

and

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} + \frac{d^2}{w} \ge \frac{(a+b+c+d)^2}{x+y+z+w}$$

and so on.

To see this, repeatedly use the two-version inequality. For instance we have

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} + \frac{d^2}{w} \ge \frac{(a+b)^2}{x+y} + \frac{c^2}{z} + \frac{d^2}{w} \ge \frac{(a+b+c)^2}{x+y+z} + \frac{d^2}{w} \ge \frac{(a+b+c+d)^2}{x+y+z+w}.$$

Challenge: Explain why we have equality,

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} + \frac{d^2}{w} = \frac{(a+b+c+d)^2}{x+y+z+w},$$

only if $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{d}{w}$.

Theorem: *For positive real numbers,*

$$\frac{a_1^2}{x_1}+\frac{a_2^2}{x_2}+\cdots+\frac{a_n^2}{x_n}\geq\frac{\left(a_1+a_2+\cdots+a_n\right)^2}{x_1+x_2+\cdots+x_n}$$

with equality only if the values $\frac{a_i}{x_i}$ *are the same.*

Setting all denominators on the left side of these inequalities equal to one completely explains puzzle 1.

> **Challenge:** Establish the famous Cauchy-Schwartz Inequality in mathematics: *For real numbers we have*
>
> $$\left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right)\geq\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2$$
>
> *with equality only if there is a constant* k *so that* $a_i=kb_i$ *for each* i.
>
> **Hint**: $a_1^2+\cdots+a_n^2=\frac{a_1^2b_1^2}{b_1^2}+\cdots+\frac{a_n^2b_n^2}{b_n^2}$.

MORE INEQUALITIES

From $\left(x-1\right)^2\geq 0$ it follows that $x+1\geq 2x$, or that $x+\frac{1}{x}\geq 2$, for x strictly positive, with equality only if $x=1$.

Here x and $\frac{1}{x}$ are two positive numbers that multiply to one. We have established

Lemma: *If* $ab=1$ *for a pair of positive numbers, then* $a+b\geq 2$.

Might the following be true?

If $abc=1$ *for a triple of positive numbers, then* $a+b+c\geq 3$.

Yes!

If $abc=1$, then at least one of the numbers is ≤ 1 and another ≥ 1. (They can't all be strictly less than 1, nor all strictly greater than 1). By reordering we can assume $a\leq 1$ and $b\geq 1$. Now, $abc=\left(ab\right)c=1$, and so $ab+c\geq 2$.

Then

$$\begin{aligned}a+b+c&=ab+c+a+b-ab\\&\geq 2+a+b-ab\\&=2-\left(1-a\right)\left(1-b\right)+1\\&=3+(1-a)(b-1)\\&\geq 3.\end{aligned}$$

> **Challenge:** Show that for positive numbers satisfying $abcd=1$, then $a+b+c+d\geq 4$.
>
> In general, argue that for positive values satisfying $a_1a_2\cdot\cdots\cdot a_n=1$ we have $a_1+a_2+\cdots+a_n\geq n$.
>
> When do we have equality?

> **Challenge:** Show that for positive numbers we have
>
> $$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}\geq n.$$

This work leads to another famous inequality in mathematics.

The Arithmetic-Geometric Mean Inequality: *For positive real numbers, their geometric mean is never larger than their arithmetic mean.*

$$\sqrt[n]{a_1a_2\cdots a_n}\leq\frac{a_1+a_2+\cdots+a_n}{n}.$$

Proof: Let G be the geometric mean as given on the left and A the arithmetic mean on the right. Then

$\frac{a_1}{G}\cdot\frac{a_2}{G}\cdot\cdots\cdot\frac{a_n}{G}=\frac{G^n}{G^n}=1$, and so

$$\frac{a_1}{G}+\frac{a_2}{G}+\cdots+\frac{a_n}{G}\ge n.$$

It follows that $A \ge G$.

The Harmonic Mean H of a set of positive numbers a_1, a_2, …, a_n is the reciprocal of the average of their reciprocals:

$$H=\frac{1}{\left(\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}\right)}.$$

The Harmonic-Geometric Mean Inequality: *For positive real numbers, the harmonic mean is never larger than their geometric mean:* $H \le G$.

Proof: Use the arithmetic-geometric mean inequality on the reciprocals.

$$\frac{1}{H}=\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}\ge\sqrt[n]{\left(\frac{1}{a_1}\right)\left(\frac{1}{a_2}\right)\cdots\cdot\left(\frac{1}{a_n}\right)}$$
$$=\frac{1}{\sqrt[n]{a_1a_2\cdots\cdot a_n}}=\frac{1}{G}.$$

It follows that $H \le G$.

Challenge: For each non-zero real number r, either positive or negative, set M_r to be the mean value

$$M_r=\left(\frac{a_1^{\ r}+a_2^{\ r}+\cdots+a_n^{\ r}}{n}\right)^{\frac{1}{r}}$$

for a given set of non-zero numbers. Let G be their geometric mean.

Show that $M_r \le G$ if r is negative and $G \le M_r$ if r is positive.

So for any set of non-zero numbers we have $H \le G \le A$. In particular, the harmonic mean of a set of positive reals is never larger than their arithmetic mean:

$$\frac{1}{\left(\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}\right)}\le\frac{a_1+a_2+\cdots+a_n}{n}.$$

Corollary: *For positive real numbers we have*

$$\left(a_1+a_2+\cdots+a_n\right)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\ge n^2.$$

Challenge: Solve puzzle 2.

Hint: Let a_i be the cost of one gallon of gloop on the i th day of the month.

FAMOUS INEQUALITIES LEADING TO *e*

Consider the sequence of numbers

$$\frac{2}{1}=2$$
$$\frac{3}{2}\cdot\frac{3}{2}=\frac{9}{4}=2.25$$
$$\frac{4}{3}\cdot\frac{4}{3}\cdot\frac{4}{3}\cdot=\frac{64}{27}\approx 2.37$$
$$\frac{5}{4}\cdot\frac{5}{4}\cdot\frac{5}{4}\cdot\frac{5}{4}=\frac{625}{265}\approx 2.44$$

and so on.

Can we prove that $\left(\frac{n+1}{n}\right)^n$ is an increasing sequence? That is, can we establish that $\left(\frac{n+1}{n}\right)^n \le \left(\frac{n+2}{n+1}\right)^{n+1}$ for all n?

Proving the hoped inequality is tantamount to establishing

$$\sqrt[n+1]{\left(\frac{n+1}{n}\right)^n} \le \frac{n+2}{n+1}.$$

The left side is $\sqrt[n+1]{1\cdot\left(\frac{n+1}{n}\right)^n}$, which, by the arithmetic-geometric inequality is

$$\le \frac{1+\frac{n+1}{n}+\frac{n+1}{n}+\cdots+\frac{n+1}{n}}{n+1} = \frac{1+n+1}{n+1}$$

just as hoped.

Challenge:

a) Show that $\left(\frac{n-1}{n}\right)^n$ also yields an increasing sequence.

b) Show that $\left(\frac{n+1}{n}\right)^{n+1}$ yields a decreasing sequence.
Hint: What is its reciprocal?

c) Examine the sequence $\left(\frac{n-1}{n}\right)^{n-1}$ for $n>1$.

These inequalities are important in the study of compound interest. We've just established that the sequence $\left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$ is an increasing sequence. It is also true that all the values of this sequence are smaller than 3. (How might one establish this?) So we have an increasing sequence that does not grow large. It follows, at least intuitively, that the sequence approaches a limit value. This limit value is called e.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
$$\approx 2.718281828459045...$$

Challenge: In a calculus one learns of a special value, also called e, which has the property that the derivative of the exponential function with that base, e^x, is just itself: $\left(e^x\right)' = e^x$. Is the number e defined via derivatives in calculus class the same number e defined via compound interest?

RESEARCH CORNER

Is it possible to explain the inequality

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \ge \frac{(a+b+c)^2}{x+y+z}$$

purely pictorially? How about

$$a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}$$

at least? (I personally can't! But I feel there must be some lovely purely visual way to see this!)

As I progressed with this essay I become less careful pointing out when our various inequalities must actually be equalities. Find the conditions that precisely determine when each inequality presented here is actually an equality.

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