A FEW THOUGHTS ABOUT ASSESSMENT

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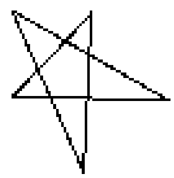
It is very easy in mathematics teaching to focus on "what" questions:

- What is measure of $\angle ABD$?
- What is the percentage increase?
- What is 357×892?
- What is the equation of the tangent line?

and miss "why" and "what if" questions:

- Why should 357×892 and 892×357 give the same answer?
- Discover something interesting about the angles in a randomly drawn five pointed star. Anything to be discovered in six- and seven-pointed stars?



- Who chose the number "360" for the number of degrees in a circle and why that number?
- Really, why is negative times negative positive?
- Is it better to use my 30% discount first and then apply sales tax, or compute sales tax first and then apply the 30% discount?
- What's the easiest way to compute 15% of 6.4?
- Why are equations of the form $ax^2 + bx + c = 0$ called "quadratics"? Where is the number four?
- Why is dividing by zero not allowed?
- Why is multiplying by 10 the same as "adding a zero at the end"?
- Is the number e one learns when studying compound interest the same as the number e one learns when differentiating exponential functions?
- Why do logarithms convert multiplications into additions? Why did Napier want this back in 1600?

There is no doubt that one needs to develop facility and ease with the basic skills of mathematics so that one is not always stumbling over small matters, but the richness and true utility of the subject comes from its conceptual structure – the problem solving and analytic tools it develops and the power of the intellectual penetration it promotes. [And as a mathematician I would add too the joyous poetic beauty this subject offers.]

ARE WE CLEAR ON OUR GOALS?

What is the ultimate drive of the upper-school curriculum mathematics experience?

- **Is it the value of the content for everyday life?**

(In which case, tell me when you last used the quadratic formula or complex numbers in your daily activities? Do we need to go beyond grade 7 or 8 for routine worldly function?)

- **Is it the practical utility of the subject for advanced applications?**

(The core standards seem to suggest much more than just this.)

- **Is it to push towards calculus? Calculus is, after all, the first baby step needed for serious scientific work.**

(In which case, why do we not begin teaching calculus concepts early on? Why wait to the senior year?)

We all understand that there is indeed human value in what we teach at the upper-school levels. But many might say that the details of the content are not actually the focus – the thinking behind them is and the process one experiences in learning how to learn something complex is key.

Mathematical thinking promotes:

- Flexible and powerful thinking
- Assuredness through systematic processing
- Perspective and connection between disparate ideas
- Clarity
- The confidence to nut one's way through problems

That is, mathematical thinking promotes life skills! We could argue that the content of the curriculum is the vehicle for learning, not itself the goal of the learning.

At the upper-school levels we work on intellectual maturity, of growth, of depth of understanding and flexibility of thought, on learning how ask questions, to create questions and to extend and push boundaries. These, after all, are the skills required for true success with innovation in business and breakthroughs in scientific research.

The Goal of Beauty: There is no doubt that mathematics possesses immeasurable utility (indeed a considerable bulk of mathematics is inspired and motivated by "real world" practical problems). But the mathematics that comes of it soon begins to address something deeper. Mankind has not engaged in mathematics for thousands of years simply because it is useful (why did the Babylonians compute hundreds of Pythagorean triples?), but because it opens to something transcendental. We should actively work to share this aspect of mathematics too! Why play the violin? Because it is beautiful. Why do math? Because it is beautiful.

RULES versus TOOLS

(A phrase coined by Anne Watson)

I have a sign over my classroom whiteboard which reads:

My ultimate goal as a teacher is a transform procedural understanding into conceptual understanding. [Conceptual understanding is so much more powerful and much more fun!]

It is my statement to the students that intellectual play focused towards conceptual understanding is the route to true success and meaning. "Getting the right answer," though important, is in some sense secondary.

This view of matters is often very unsettling to students. Many have been taught, and rewarded, to focus on getting the right answer - quickly. Textbooks give the impression that this is the goal:

> *Answer these questions that you yourself have not asked and get the same answers listed at the back. Don't worry – these are not new questions – someone else has answered them all already. Be sure to do at least 40 of them for homework tonight. (And while you are at it, do some push-ups.)*

In order to succeed with the "get the right answer quickly" approach students often do that which is utterly appropriate for that goal: Memorize rules that lead to the right answers.

- To multiply by 10 add a zero.
- To divide by a fraction multiply by the reciprocal
- The vertex is at $x = -b/2a$
- $|x-a| > b$ is OR; $|x-a| < b$ is AND.
- The derivative of a^x is $\ln a \cdot a^x$
- FOIL

There is now doubt that mathematicians use these quick rules too – when I multiply by 10, I do indeed just "add a zero at the end" – but mathematicians work from a base of understanding first. I know, for example, that when working in base two to multiply by 2 "adding a zero at the end" is the appropriate thing to do. (Is that obvious?)

So an alternative version of my classroom whiteboard statement would be:

Let's turn rules into tools.

ON ROUTINE ASSESSMENT

Don't get me wrong – routine exercises and practice problem sets on skills and procedures have their place. One must practice ideas and make lots of small errors (and catch them!) when learning a sequence of new ideas. Traditional homework assignments, graded pieces, quizzes and tests are an integral part of the school room experience. But one can ask ..

- Do quizzes and tests need to be timed? To what extent does it matter how long a student takes to complete a set of problems?
- Can some quizzes be done in pairs?
- Can some quizzes or exams be open-note exams?
- Can students grade each others' work?
- Can students be asked to submit possible questions for an upcoming exam? [Teaching the art of writing good questions is worthwhile pursuit!]
- Can students give answers orally? Up on the board?

and so on. (I am sure you can think of many more possibilities to add to this list.)

A BASIC, but perhaps profound, QUESTION: Do we need to give numerical scores to quizzes and assignments? Imagine that, akin to the role of an English teacher, your role as an educator really was to assess a students' ability to synthesize and process ideas, to communicate answers well and be on a very good path to getting it right. How would that free you as an educator re the types of questions you might ask: "Explore this idea and come up with something interesting?" "Do you think this claim is true?" "What do you think of this answer to this question?" "Is this proof correct or is there an error in it somewhere?"

How would a lack of numerical scores change the student mindset of what the expectations are in a math class?

If your students have the appropriate self-reliance and maturity one can consider handling chapter homework problems along the following lines.

- Do enough of chapter 21 problems until you feel you understand what is going on
- Of the ones left, pick the one you feel will be the easiest to do and the one you feel would be the hardest to do. Do them, and see if your predictions were right. Write me a sentence or two about this experience. (Be sure to let me know which two problems you worked on.)
- Of the hard questions you looked at what makes the hardest ones hard? Make up one problem that is similar to these hard questions to give to a friend, one that uses the same ideas but is a bit easier to solve.

COMMENT: In her essay "Adolescent Learning and Secondary Mathematics" (Proceedings of the 2008 Annual Meeting of the Canadian Mathematics Education Study Group/ Groupe canadien d'etude en didactique des mathematiques, 21-29), Anne Watson offers similar and further ideas of this type.

I routinely give two types of quizzes that sometimes shock folk when they learn I do this:

A 100% PACKET : This is a fairly large problem set which students must complete over a period of three or four weeks and obtain a perfect score of 100% on it! They may use their notes, they may consult with me, and they are expected to hand in the packet multiple times for partial grading and feedback. They must keep working at it until they have a perfect score. (Otherwise it is a score of 0%.)

A QUIZ WITH ALL THE ANSWERS SUPPLIED: I often give a quiz with all the answers supplied. (A question might appear: *A chord in a circle of radius 10 subtends an arc of 42 degrees. What is the length of the chord? [The answer is 24.3]*) This reinforces the point that I am not so much interested in the numerical answer, but the method towards obtaining it. (It also adds a psychological comfort for students knowing that their answers are correct.)

META-ASSESSMENT

In order to promote true conceptual understanding, one can ask "meta-questions." These come in a number of styles:

- Spot the Error
- The Head-On Approach (for battling classic errors)
- Explain
- Think Before You Leap questions
- Discover and Explore
- Jolt!
- Pushing the Boundaries

Some examples:

1. SPOT THE ERROR

Question: In answering the question: XXXXXXX
Lizzy wrote: XXXXXXX

Why did the teacher give her a score of only 3/5 for her response?

Question: Consider the following geometry proof question: XXXXXXX
Gerald wrote:

XXXXXXX

But there is a mistake in Gerald's work that causes his proof to logically collapse! What is Gerald's error?

Can you write a logically sound proof for this question?

Question:

a) A student writes in his homework: $6x = 18 = x = 3$. What does this actually say? What do you think the student was trying to say?

b) A student writes in her homework: $\frac{3(x+2)}{5} = \frac{3x+6}{5} = \frac{3}{5}x + \frac{6}{5}$. Is this saying something reasonable do you think?

c) A student writes in his homework:

$$2(x-2) = 1 + x$$
$$= 2x - 4 = 1 + x$$
$$= x = 5$$

Do you think this is saying something reasonable? Any advice for the student?

d) A student is asked to expand $x(x-2)$ and writes $x = 0, 2$. What do you think the student was doing? Any advice?

2. TAKING CLASSIC ERRORS HEAD-ON

Battling with how hard it is to make something that is generally false work often cements the idea that it really is resistant to working!

Question: Many students like to think that $(a+b)^2 = a^2 + b^2$.

a) Choose some specific values for a and b to show that this is not true in general.
b) Find a value for a and a value for b for which, by luck, $(a+b)^2 = a^2 + b^2$ happens to hold.
c) Use algebra to find all values for a and b for which $(a+b)^2 = a^2 + b^2$ happens to be hold.

One can apply this type of question to all the classic algebraic errors:

$$\sqrt{a^2+b^2} = a+b \qquad \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} \qquad 3 \cdot \frac{a}{b} = \frac{3a}{3b} \qquad \frac{2x+y}{2z} = \frac{x+y}{z}$$

Question: In answering the question:

Find the equation of the tangent line to the curve $y = x^3 + 5x^2$ at the point $(1,6)$

Poindexter wrote:

$$y - 6 = 3x^2 + 10x(x-1)$$

What do you think of his answer?

Question: The height of a star at an angle of elevation 30° is the same as the height of a start at 150°. Thus the following is true:

$$\sin\left(30^\circ\right) = \sin\left(150^\circ\right)$$

Lulu says: *Divide both sides by sin and get* $30 = 150$ *.*

$$\cancel{\sin}\left(30^\circ\right) = \cancel{\sin}\left(150^\circ\right)$$
$$30 = 150$$

But clearly 30 does not equal 150!

How would you explain to Lulu that what she is doing is wrong?

Question: Gordie thinks that the following is a valid log rule:

$$\log M \cdot \log N = \log\left(M + N\right)$$

He says that it turns multiplications into additions, which is indeed what logarithms do.

BUT HE IS NOT CORRECT!

What is the correct version of the log rule that Gordie is trying to write down? How would you advise him to think about the log rule so that it will be clear in his mind what the correct version should be?

3. EXPLAIN

Question: Joyce says that the value of $5^{\log_5 37}$ is obvious if you think about it. And she is right!

What is the value of this quantity and how would you explain to Quentin, who doesn't "get it"? He doesn't see why the answer is what it is.

Question: Nervous Nelly, who prefers to memorise "rules" for mathematics, was once told that multiplying an inequality by a negative number "flips" the inequality.

For example, if $C < D$ then $-6C > -6D$. And if $x < -3$, then $-2x > 6$.

a) Has she memorized a correct rule?
b) Nelly admits she does not understand why the rule she memorized is true. How would you explain its validity to her?

Question:

a) Work out $\frac{12}{15} \div \frac{3}{5}$ and show that it equals $\frac{4}{3}$.

b) Now notice that

$$12 \div 3 = 4$$
$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3}$$

Is this a coincidence or does $\frac{a}{b} \div \frac{c}{d}$ always equal $\frac{a \div c}{b \div d}$?

4. THINK BEFORE YOU LEAP

Question: Here are four quadratic equations:

(A) $y = 4(x-3)(x-7)$
(B) $y = 3(x-2)^2 + 6$
(C) $y = 2x^2 - 4x + 8$
(D) $y = x^2 + x(x-3)$

i) For which equation would it be easiest to answer the question: *What is the vertex of the quadratic?*
ii) For which equation would it be easiest to answer the question: *Where does the quadratic cross the x-axis?*
iii) For which equation would it be easiest to answer the question: *What is the smallest value the quadratic adopts?*
iv) For which equation would it be easiest to answer the question: *What is the line of symmetry of the quadratic?*
v) For which equation would it be easiest to answer the question: *What is the y-intercept of the quadratic?*

Question: For each of the following describe an easy way to compute these values without a calculator. Either describe the method in words or write a line of arithmetic that illustrates a good way to proceed.

a) 82×5 b) $35 \times 35 \times 40$ c) 7×16 d) 198×32

e) $87 \cdot 903 + 13 \cdot 903 + 17$

f) $196 - 37$ g) $817 - 69$

g) 621 divided by 5

h) 15% of 62

i) $\frac{13}{66} \cdot \frac{33}{28} \cdot \frac{7}{13}$

j) $603 \div 97$ k) $813 \div 198$

Question: Which of the following problems is <u>not</u> easy to work out in your head?

$$23\times 37-13\times 37$$
$$27\cdot 153+73\cdot 153$$
$$3(7)+87(7)$$
$$105(105)-95(105)$$
$$17\times 13+13\times 3$$
$$34\times 7+34\times 6$$

Question: Compute $(7-18)(8-18)(9-18)(10-18)\cdot\cdots\cdot(25-18)(26-18)$.

Question: Evaluate:

$$\frac{\frac{4-2}{8-3}-2\cdot\frac{7-(12-6)}{30-\frac{5\times 5}{4-3}}}{\frac{19}{3}+\frac{17}{2+\frac{1}{\left(17-\frac{4}{5}\right)}}}$$

Question: Which of the following statements seem they could be true? Which are definitely wrong?

(Answer this question without actually computing the products. Not one of them is actually correct! This is an exercise in estimation only.)

$$999\times 31=30999$$
$$12\times 198=1996$$
$$106\times 213=206,816$$
$$9458\times 9786=192837261748$$
$$19990\times 4=76987$$

Question:

a) How big an answer do you expect from computing 19×998?
b) Would computing 5671×1772 give an answer in the millions?
c) Would computing 1123×1005 give an answer in the millions?
d) Would 997×998 give an answer in the millions?

Question: Here are four problems. Don't answer them(!), but tell me what type of problem each is. Is it about

ADDITION AND SUBTRACTION of fractions

Or

MULTIPLICATION of fractions

Or

DIVISION of fractions?

(A) Tom's driveway is $\frac{1}{4}$ of a mile long. If Tom walks at a speed of three-and-a-half miles per hour, how long will it take him to walk the length of his driveway?

TYPE =

(B) One third of a field is planted with corn, one quarter with cabbages, and the rest with squash. What fraction of the field is planted with squash?

TYPE =

(C) John earns $3500. He gives 30% of income to the IRS and the one third of what remains to his mother. How much money does John have remaining?

TYPE =

(D) Alfred purchases a suit, normally priced at $315 but was on sale for a 20% discount. If the state charges 8% sales tax, how much did Alfred end up paying for his suit?

TYPE =

Question: Quickly … solve:

a) $(x-2)(x-14)(x-22)=0$

b) $(x+1)^2=25$

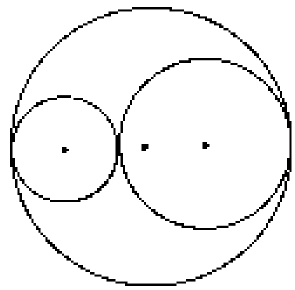
Question: A parabola passes through the points $(2,5)$, $(3,-6)$ and $(10,5)$. What is the x-coordinate of its vertex?

Question: Find $\lim_{h\to 0}\frac{(3+h)^2-81}{h}$.

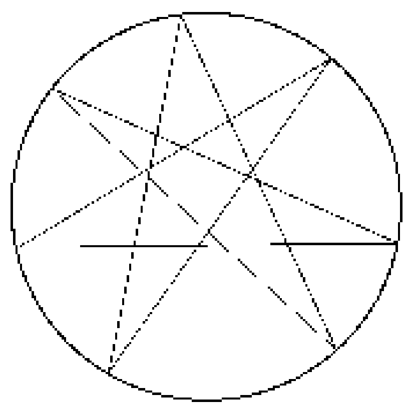
Question: Find the area of a triangle with side lengths 9 inches, 8 inches and 19 inches.

5. DISCOVER AND EXPLORE

Question: The centers of all the circles in this picture are collinear. Discover and explain something interesting about the circumferences of these circles



Question: Discover and explain something interesting about the angles of a seven-pointed star drawn inside a circle as shown.



Question: Playing on a calculator Pandi noticed that 2^{46}, 2^{56} and 2^{76} each begin with a seven.

a) Find the next few powers of two that begin with a seven.
b) Explain why the pattern you are seeing stops.

Question: Sketch $y = x^{\frac{1}{\ln x}}$ on a calculator. Explain!

6. JOLT!

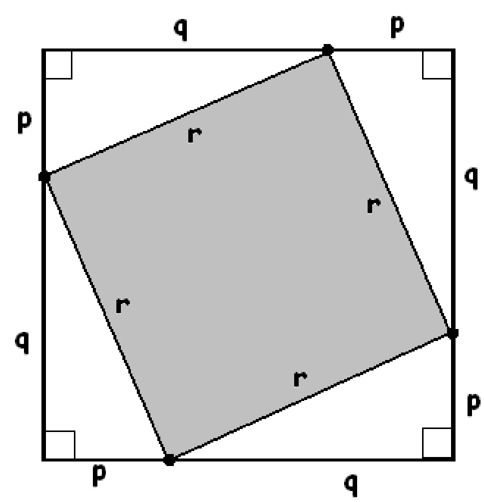
Bring in unexpected connections that provoke clarity!

Question:
a) Compute $(x^6 + x^5 + 5x^4 + 5x^3 + 9x^2 + 5x + 2) \div (x^2 + x + 2)$
b) Put $x = 10$ into the problem of part a). What grade-five division problem have you just solved? Does your answer seem to be correct?

People forget in an algebra course that x can actually be a number!

Question:
a) Show that $x^5 - 1$ is divisible by $x - 1$.
b) Is $2^{100} - 1$ prime?

Question: Compute the area of the central shaded square two different ways:



Question: Consider the differential equation $\frac{dy}{dx} = iy$ with $y(0) = 1$.

a) What would be the standard solution to this differential equation?
b) Show that $y = \cos x + i \sin x$ is also a solution.

What might you be tempted to now say?

7. PUSHING THE BOUNDARIES

Question: a) Explain why the following party trick works:

> *Think of two single-digit numbers.*
> *Take the first digit and multiply it by 5.*
> *Add 5 and then multiply the result by 4.*
> *Add the second number you thought of and subtract 20.*
> *Add the second number you thought of a second time, and then halve the result.*
> *You are now thinking of a two digit answer.*
> *Your answer is composed of the two digits you first thought of!*

b) Make up your own party trick that involves thinking of three single digit numbers to produce a result that is a three-digit number composed of those three initial digits.

Question: In computing 654 + 179 Iggy writes:

```
  654
+ 179
-----

7 12 13   = 833
```

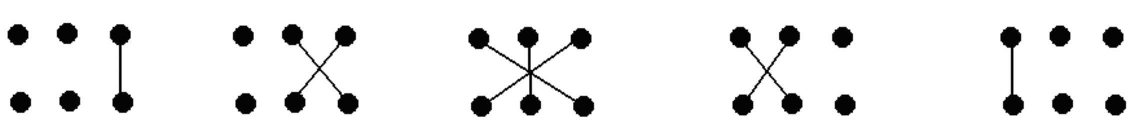
Does this represent valid mathematical thinking? Briefly explain what you would guess Iggy was thinking.

Question:

a) When asked to find a fraction between $\frac{1}{12}$ and $\frac{1}{11}$, Poindexter wrote: $\frac{1}{11\frac{1}{2}}$. Is this a mathematically valid answer?

b) Quickly write down three fractions that lie between $\frac{1}{12}$ and $\frac{1}{11}$.

Question: Vedic mathematics taught in India (and established in 1911 by Jagadguru Swami Bharati Krishna Tirthaji Maharaj) has students compute the multiplication of two three-digit numbers as follows:



What do you think this sequence of diagrams means?

8. SOMETHING IS NOT RIGHT

These are like "spot the error" questions but require a deeper level of thought.

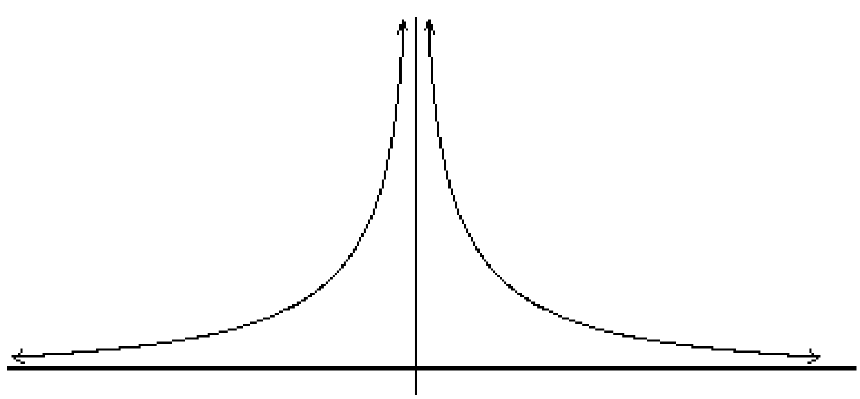
Question: Could $56452 \times 18863 = 98611987364$ be correct?

Question: My calculator says that $\sqrt{46}$ equals 6.782329983. Why can't this be correct?

Question: Could the sum of 19000 odd numbers end with a five?

Question: Explain why the formula $(a+b)^4 = a^4 + b^4 + 4ab^3 + 6ba^3 + 4a^2b^2$ cannot be right.

Question: When I plot the curve $y = \frac{200}{1+x^4}$ on my calculator I see a vertical asymptote at $x = 0$. Do you? Should you?



Question: Sketch the graphs $y = x^2$ and $y = 2^x$ simultaneously on a calculator to se they intersect twice. Is this right? Do they intersect exactly two times?

Question: Tatiana says that $10! = 8542677640$. Quickly, why must she be mistaken?