An Essay on a

GRID-MOVING PUZZLE

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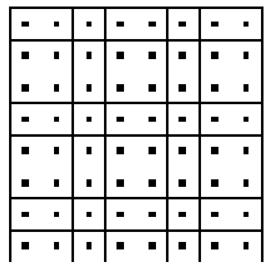
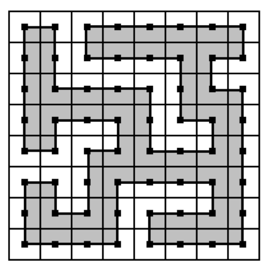
A CLASSIC PUZZLER

Here is a 8×8 grid of squares with center points highlighted. Assume each cell has unit area.



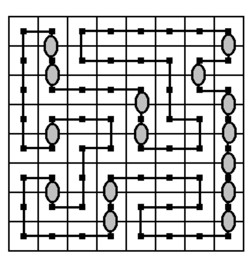
Starting in one cell, draw a path returning to start that visits each center-point exactly once taking unit steps left, right, up or down only (no diagonal steps). Here's one example of such a path. Draw a different example of your own.



My example produced a polygon of area 31. I bet your example did too.

CHALLENGE 1: Explain why all polygons produced this way must have the same area.

In my example I took 15 steps in the upward direction ($U=15$).



I also took 15 down steps, $D=15$, 17 right steps, $R=17$ and 17 left steps, $L=17$.

These are all odd numbers with $U=D$, $R=L$ and $U\neq R$. I bet in your example you too have two distinct pairs of odd numbers for these counts.

CHALLENGE 2: Why must number of steps in any one direction be odd? Why must $U = D$ and $R = L$? Why can't U equal R?

Why focus on 8×8?

CHALLENGE 3: Why is it impossible to draw a path returning to start that visits each and every center point in a 7×7 grid of squares? (Again, left, right, up, down steps only.)

COMMENT: In fact, no such loop exists on any odd-by-odd grid. One can't even begin this puzzle on these odd grids.

CHALLENGE 4: Show that it is possible to draw a path returning to start on a 6×6 grid of squares. Prove that U, D, L, R are again sure to be odd numbers. Find an example of a path with U, D, L, R all equal.

THE ANSWERS:

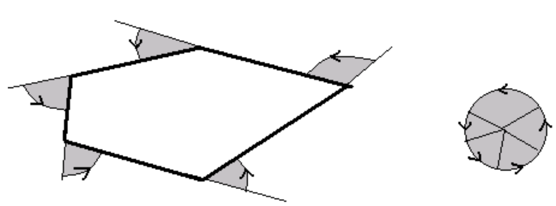
This is such a tricky puzzler. It takes quite some doing to establish the claims made.

We need some observations:

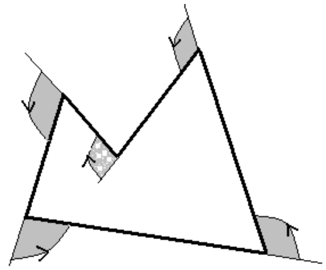
1. Any path of steps in a square grid that returns to start must have an equal number of up and down steps (otherwise one shan't return to the same height), and an equal number of left and right steps. (This explains part of CHALLENGE 2). Thus the total number of steps in <u>any</u> loop on a grid must be even:

$$R + L + U + D = 2R + 2U$$

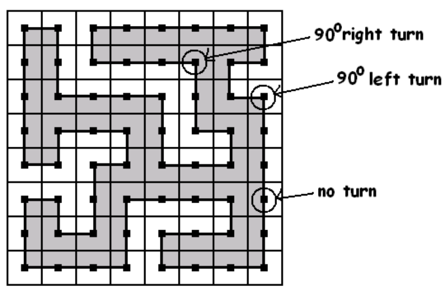
2. One learns in high school geometry that the exterior angles of any convex polygon amount to one full turn of rotation.



This remains true even for concave shapes: clockwise motion negates counter-clockwise motion but the net effect is still one full turn.



3. Consider a polygon created from walking on a square grid. Suppose we walk in a direction so that the interior of the polygon always lies to our left. For each step we either continue in the same direction, turn left 90°, or turn right 90°. As the exterior angles must amount to one full turn overall, the number of 90° left turns one takes is 4 more than the number of 90° right turns.

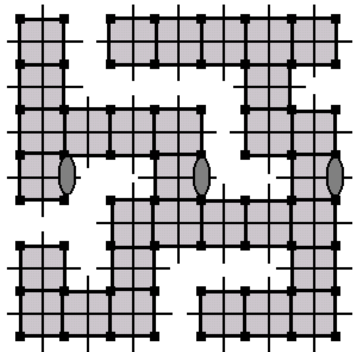


If the interior of the polygon is shaded, $\frac{1}{4}$ of a square cell is shaded at each left turn, $\frac{3}{4}$ at each right turn, and $\frac{1}{2}$ of a cell at all other places. One average, half of each square cell is shaded EXCEPT for the excess of four left turns each missing $\frac{1}{4}$ of the area needed to maintain that average. Thus the area of the polygon is for sure $\frac{1}{2}\cdot 64 - 4\times\frac{1}{4} = 31$.

COMMENT: In general, for a grid with an even number of rows or even number of columns (or both), the area of the polygon formed is always one less than half the area of the grid.

COMMENT: This result also follows readily from Pick's Theorem: *The area of any lattice polygon is* $I + \frac{1}{2}B - 1$. In our case, the number of interior points I is zero, and the number of boundary points B equals the number of cells in the grid.

4. Imagine the polygon itself divided into unit squares. (We've kept the original 64 center points visible.)



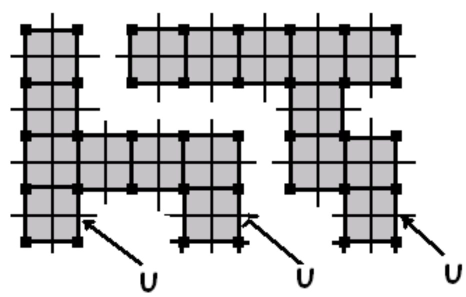
I claim:

> **On any row, the number of shaded cells plus the number of up steps on that row is even.**

For example, on the fourth row from the top shown, there are 3 shaded cells and 3 up steps: $3+3=6$. On the bottom row we have $6+2=8$.

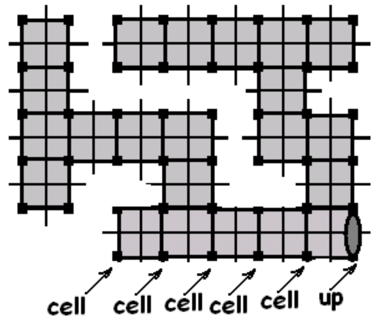
Let's focus on the fourth row to see why this is so.

Draw the top four rows of the polygon.



Here the polygon appears as separate loops. By observation 1, each loop involves an even number of steps, and so an even number of the original center points. All eight center points are used at each horizontal level, except at the bottom of the fourth row. Since the total number of center points is even, this means that there are an even number of center points at the bottom of truncated picture. But notice, each of center point at the bottom level either sits as the lower left corner of a cell on the fourth row or as the bottom of an up step on the fourth row. This sum must be even.

As another example, for the fifth row we see 5 cells and 1 up step.



So at each row we have:

#shaded cells + #ups = even

Adding up all eight rows gives:

Total number of shaded cells. + total number of ups = even.

But the total number of shaded cells is the area, which is 31, and so the total number of Us is odd. In the same manner R is also odd.

So we have $U = D$ and $R = L$ with odd numbers. All four numbers cannot equal the same value as 64, the total number of steps (one for each center point) is not the sum of four equal odds. This finishes CHALLENGE 2. Phew!

For CHALLENGE 3: Colour the grid of cells red and blue like a checkerboard. If each dimension of the board is odd, there is one more red cell than blue cell. But any loop that walks from cell to cell visits cells of alternating colours and has an even number of steps and so requires an equal number of red and blue cells. This mismatch of counts shows that there can be no loop that visits every cell of an odd-by-odd grid.

For CHALLENGE 4: The work we've conducted applies to any grid of squares with an even number of rows or even number of columns (or both). (Why can we be sure loops that visit each and every cell exist?) If the total count of squares is a multiple of eight, as for the 8×8 grid with 64 squares, the numbers U, D, R, L cannot be the same odd number. However, if the total count of squares is not a multiple of eight, such as the 6×6 grid with 36 squares, it is possible to for U to equal R. Constructing specific examples is not hard.

RESEARCH CORNER: Can anything be discovered about areas of loops on triangular grids? Anything about the number of steps in a given direction?