INTEGER TRIANGLES

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The following represents a sample activity based on the May 2006 newsletter of the St. Mark's Institute of Mathematics (www.stmarksschool.org/math).

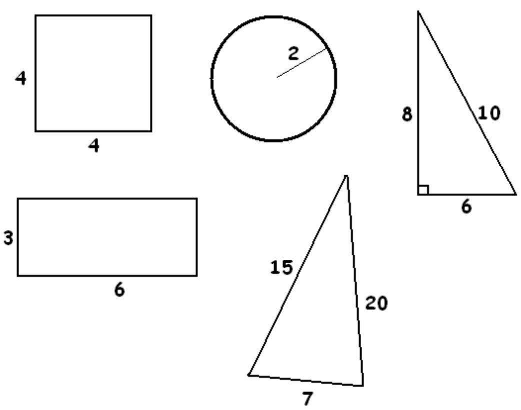
This essay is a draft chapter from a forthcoming book.

The St. Mark's **Institute of Mathematics** Newsletter

MAY 2006

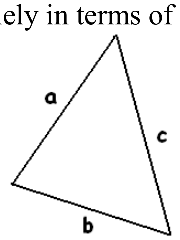
THIS MONTH'S PUZZLER: MORE THAN JUST AN AREA!

What curious property do each of the following figures share?



Find another right triangle (with integer sides) with this property. Is there another integer rectangle with this property?

HERON'S FORMULA: In 100 C.E. Heron of Alexandria (also known as Hero of Alexandria) published a remarkable formula for the area of a triangle solely in terms of its three side-lengths:



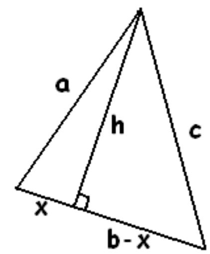
$$\text{area} = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$

Thus the area A of the 15-20-7 triangle above is:

$$A = \frac{1}{4}\sqrt{(42)(2)(28)(12)} = \frac{\sqrt{28224}}{4} = 42.$$

Proving Heron's formula is not difficult conceptually. (The algebra required, on the other hand, is a different matter!) Here are two possible approaches:

Proof 1: Draw an altitude and label the lengths x, $b-x$, and h as shown:

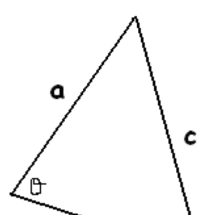


We have:

$$(b-x)^2 + h^2 = c^2$$
$$x^2 + h^2 = a^2$$

Subtract these two equations and obtain a formula for x. Substitute back in to obtain a formula for h. Now use $A = \frac{1}{2}bh$ (and three pages of algebra) to obtain Heron's formula.

Proof 2: We have $A = \frac{1}{2}ab\sin\theta$.



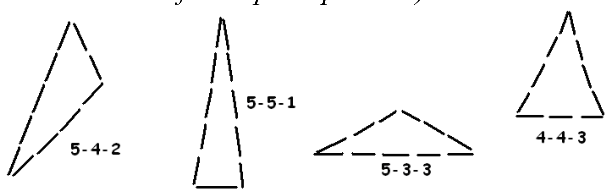
By the law of cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$
Solve for $\sin\theta$ and for $\cos\theta$ and substitute into the identity:

$$\cos^2\theta + \sin^2\theta = 1.$$

Out (eventually!) pops Heron's formula.

MATCHSTICK TRIANGLES
Here's a great student challenge/activity:

With 11 toothpicks it is possible to make ***four*** *different triangles of perimeter 11(using a whole number of toothpicks per side):*



(Why isn't 6-3-2 a valid triangle?)

Surprisingly the count goes down if one adds another toothpick to the mix: With 12 toothpicks one can only make ***three*** *different triangles: 5-5-2; 5-4-3; 4-4-4.*

What's going on?

Construct a table showing the number of integer triangles one can make with 1, 2, …, 20 toothpicks. What do you notice about the even and odd entries? Do you see any patterns?

In 2005, high-school (and some younger) students of the St. Mark's Research Group played with this problem and discovered—and proved—the following remarkable formula:

The number of triangles one can make with N toothpicks is:

$\left\langle \frac{N^2}{48} \right\rangle$ *if N is even* and $\left\langle \frac{(N+3)^2}{48} \right\rangle$ *if N is odd.*

(The angled brackets mean "round to the nearest integer.")

Thus one can make $\left\langle \frac{100^2}{48} \right\rangle = 208$ different triangles with 100 toothpicks!

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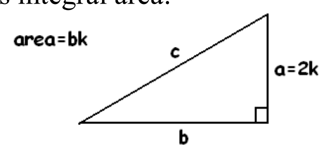
RESEARCH CORNER:
GOING FOR INTEGERS ALL ROUND

The 15-20-7 triangle on the previous page shows that it is possible for a triangle to have integer side-lengths and integer area.

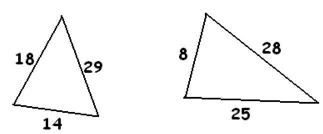
There many examples of such triangles. For example, any right triangle with one leg an even integer has integral area:



But the 15-20-7 triangle has the added property that, not only are its side-lengths, perimeter, and area are all integers, the perimeter and the area have the same integer value! (namely, 42).

Challenge 1: Find another (non-right) integer triangle with perimeter equal to area.

Another idea … It is possible for two different integer triangles to have the same perimeter and the same area. The following two triangles are an example of such a pair.

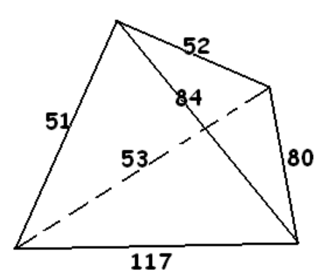


Alas, the area of each triangle here is not an integer.

Challenge 2: Does there exist a pair of integer triangles with the same perimeter and the same integer area?

Challenge 3: Let T(N) be the number of integer triangles with perimeter N and integral area. Is there a formula for T(N) akin to the formula for toothpick triangles?

Just for fun … Here's an integer tetrahedron with each face of integer area and volume also an integer!



INTEGER TRIANGLES
May 2006

COMMENTARY, SOLUTIONS and THOUGHTS

Each shape in the opening puzzler of the newsletter has the property that the numerical value of its perimeter equals the numerical value of its area. (Is one allowed to say "perimeter equals area"?)

There are only two integer rectangles with this property.

> Suppose a rectangle with sides of lengths a and b has area matching perimeter. Then $ab = 2a + 2b$ yielding:
> $$b = \frac{2a}{a-2} = 2 + \frac{4}{a-2}$$
> For the right to be an integer we require $a-2$ to have value 1, 2, or 4. Thus only the 3×6 rectangle and the 4×4 square have the desired property.

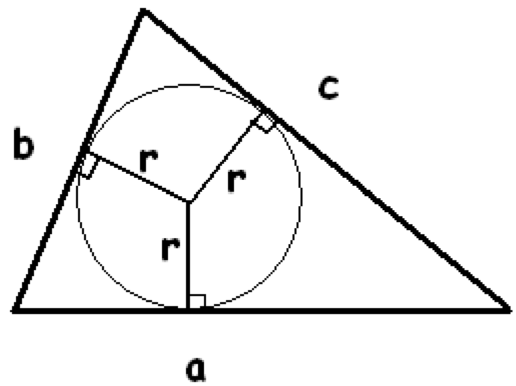
COMMENT: As observed by high school teacher Michael Ericson, the condition $ab = 2a + 2b$ can be rewritten: $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$. (Read on!)

There are only two integer right triangles with this same property:

> Suppose a right triangle, with legs of integer lengths a and b, has area matching perimeter. Then $\frac{1}{2}ab = a + b + \sqrt{a^2 + b^2}$. Squaring gives:
> $$a^2 + b^2 = \left(\frac{1}{2}ab - a - b\right)^2 = \frac{1}{4}a^2b^2 + a^2 + b^2 - a^2b - ab^2 + 2ab$$
> yielding:
> $$a^2b + ab^2 = \frac{1}{4}a^2b^2 + 2ab$$
> Dividing through by ab and solving for b produces:
> $$b = \frac{4a-8}{a-4} = \frac{4a-16+8}{a-4} = 4 + \frac{8}{a-4}$$
> For the right to be an integer, a must adopt one of the values 5, 6, 8, or 12. This shows that the 6-8-10 and 5-12-13 triangles are the only two integer right triangles with the desired property.

If we let go of the requirement that the side lengths of the triangle must be integral, we can say, in general:

> *The perimeter and area of a triangle have the same numerical value if, and only if, the inradius of the triangle is* 2.



Following the notation suggested by the diagram we see that the area A and perimeter P of a triangle satisfy:

$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}Pr$$

(The same relation holds for any polygon with sides tangent to a common circle.)

COMMENT: The condition that the inradius r be two can be written: $\frac{1}{r} = \frac{1}{2}$.

CURIOUS CHALLENGE:

a) Show that a parallelogram has area equal to perimeter if, and only if, $\frac{1}{h_1} + \frac{1}{h_2} = \frac{1}{2}$ where h_1 and h_2 are the two heights of the figure.

b) Show that the area of a triangle equals its perimeter if, and only if, $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{2}$ where h_1, h_2 and h_3 are the three altitudes of the triangle.

c) Show that an $a \times b \times c$ rectangular prism has volume equal to its surface area if, and only if, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$.

d) Show that a right circular cylinder has volume equal to surface area if, and only if, $\frac{1}{r} + \frac{1}{h} = \frac{1}{2}$ where r is the radius of its base and h is its height.

e) Show that a regular circular cone has volume equal to surface area if, and only if, $\frac{1}{r}+\frac{1}{b}=\frac{1}{2}$ where r is the radius of its base and b is the perpendicular distance of the center of its base to its lateral surface.

f) Is something interesting going on here? Can one associate with any geometric figure some "fundamental" set of (perpendicular?) measurements that dictate the properties of its area and perimeter or, in three dimensions, its volume and its surface area? Is it problematic that for a sphere volume equals surface if, and only if, $\frac{1}{r}=\frac{1}{3}$? (Or would it be more appropriate to write $\frac{1}{d}+\frac{1}{d}+\frac{1}{d}=\frac{1}{2}$ where d is the diameter of the sphere?)

See the comment at the end of this essay for a few thoughts on this.

PAIRS OF INTEGER TRIANGLES OF EQUAL AREA AND EQUAL PERIMETER

In the newsletter I presented a pair of integer triangles possessing the same perimeter and the same area:

The 14-18-29 *and* 8-25-28 *triangles each have perimeter 61 and area* $\frac{5\sqrt{3111}}{4}$.

As is often the case with problems of this ilk, if one example exists, infinitely many do! Tyler Jarvis of Brigham Young University pointed me in the right direction to hunt for other solutions.

The challenge is to find integer solutions to the equations:

$$a+b+c=d+e+f$$
$$s(s-a)(s-b)(s-c)=s(s-d)(s-e)(s-f)$$

where $s=\frac{a+b+c}{2}$. (Here a, b, c and d, e, f represent the side-lengths of the two triangles.) Eliminating the variable f and simplifying means we seek integer solutions to the single equation:

$$\begin{aligned}&(-a+b+c)(a-b+c)(a+b-c)\\&\quad=(a+b+c-2d)(a+b+c-2e)(2d+2e-a-b-c)\end{aligned} \qquad (*)$$

Actually, seeking rational solutions will suffice. (Any rational solution can be converted to an integer solution by multiplying through by a common denominator.)

Let S denote the set of all rational points (a,b,c,d,e) that satisfy the required equation (*).

We certainly have within S a whole host of trivial solutions, namely, those that correspond to the two triangles being identical: $d=a$ and $e=b$ (and consequently $f=c$). Let A be any one of these basic solutions:

$$A=(a,b,c,a,b)$$

We also have the one particular solution:

$$P=(14,18,\ 29,8,\ 25)$$

Our strategy is to look at the line that connects the points A and the point P and see if it again intersects the set S. For each real number λ, set

$$\begin{aligned} X_\lambda &= \lambda A+(1-\lambda)P \\ &=(\lambda a+14(1-\lambda),\lambda b+18(1-\lambda),\lambda c+29(1-\lambda),\lambda a+8(1-\lambda),\lambda b+25(1-\lambda)) \end{aligned}$$

For this point to lie in S its components must satisfy the equation (*). This yields a cubic formula in λ. We know that $\lambda=0$ and $\lambda=1$ are solutions to this cubic, and so we can factor from it λ and $\lambda-1$ to yield a linear equation in λ with rational coefficients. This yields a third rational solution to (*). One still needs to check that this solution corresponds to a meaningful geometric solution (namely, the sides of the triangle are all positive and the triangular inequalities hold so that the triangle actually exists!) but this approach does get us beyond the most difficult part of the search: finding examples of numbers that satisfy (*).

With a short computer program one can indeed generate many examples of integer triangle pairs with equal areas and equal perimeters. Here are a few:

$10-34-39$ and $19-24-40$
$18-27-30$ and $20-24-31$
$35-57-62$ and $42-47-65$
$40-61-66$ and $46-52-69$
$45-94-94$ and $49-84-100$

COUNTING INTEGER TRIANGLES

Students of the St. Mark's Institute Research class played with this challenge in 2005 ([TANTON]), as had the 2002 students of the Boston Math Circle ([FOCUS]).

Counting the number of distinct triangles one can form with $n=1,2,3,\ldots$ toothpicks yields the sequence of numbers:

0, 0, 1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, …

Let's denote the n term of this sequence $T(n)$. One notices after a moment that this list is composed of two intertwined copies of the same sequence of numbers 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, … , with one copy shifted three places. That is, it seems:

$$T(2n) = T(2n-3) \text{ for } n > 1$$

If this is indeed true, we need then only consider triangles with even integer perimeter in our pursuit of a formula for these counts.

Let's take a moment to collate some facts about integer triangles.

1. Three positive integers a, b and c (written in non-decreasing order) are the side-lengths of a triangle if, and only if, $a+b>c$.

This is the triangle inequality.

2. No integer triangle (or any triangle for that matter) possesses a side of length greater than or equal to half the perimeter of the triangle.

The remaining two sides would sum to a value less than half the perimeter and this violates the first point.

3. No integer triangle with even perimeter has a side of length 1.

This follows from point 2.

4. If a, b and c (written in non-decreasing order) are the sides of an integer triangle of even perimeter, then $a+b>c+1$.

We certainly have $a+b>c$. If $a+b=c+1$, then its perimeter $a+b+c$ would be odd.

5. If a, b and c are the sides of an integer triangle of even perimeter $2n$, then $a-1$, $b-1$ and $c-1$ are the sides of a valid triangle of (odd) perimeter $2n-3$. And conversely, adding one to the value of each side of a triangle of perimeter $2n-3$ produces a valid triangle of perimeter $2n$.

One checks that the necessary inequalities hold.

Point 5 shows that we have a perfect matching between the triangles of perimeter $2n$ and the triangles of perimeter $2n-3$ and so, indeed, $T(2n) = T(2n-3)$ for all $n>1$.

In pursuit of a formula for $T(n)$, the 2005 students of the St. Mark's Institute class made the following key discovery:

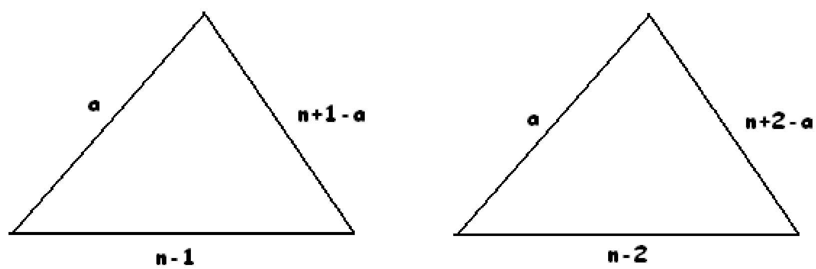
Lemma: *For N even with $N > 12$ we have* $T(N) - T(N-12) = \frac{N}{2} - 3$.

Proof: Write $N = 2n$.

Let $a \le b \le c$ represent the sides of an integer triangle of perimeter $N-12$. By point 2, c is at most $n-7$. Then $a+4$, $b+4$ and $c+4$ represent the side lengths of a valid triangle of perimeter N with longest side at most $n-3$. The correspondence

$$(a,b,c) \leftrightarrow (a+4, b+4, c+4)$$

provides a perfect matching between these two types of triangles, but it omits the triangles of perimeter N with longest side $n-2$ or $n-1$. How many of these triangles is this correspondence overlooking?



The triangle inequality shows that, for a triangle of perimeter $2n$ and longest side $n-1$, the shortest length a of the triangle satisfies $2 \le a \le \frac{n+1}{2}$, and for a triangle of perimeter $2n$ and longest side $n-2$ the shortest side a satisfies $4 \le a \le \frac{n+2}{2}$. There are thus a total of $\left\lfloor \frac{n+1}{2} \right\rfloor - 1 + \left\lfloor \frac{n+2}{2} \right\rfloor - 3 = n-3$ of these triangles. Consequently $T(N)$ is larger than $T(N-12)$ by a count of $n-3$. This establishes the lemma. $\square$

Define $T(0)$ to be zero. If we write $N = 12k + r$ with $r = 0$, 2, 4, 6, 8 or 10, then:

$$
\begin{aligned}
T(N) = {} & T(N) - T(N-12) \\
& + T(N-12) - T(N-2\cdot 12) \\
& + \cdots \\
& + T(12+r) - T(r) \\
& + T(r)
\end{aligned}
$$

$$
\begin{aligned}
& = \frac{12k+r}{2} - 3 \ + \ \frac{12(k-1)+r}{2} - 3 + \ \cdots + \frac{12\cdot 1 + r}{2} - 3 + T(r) \\
& = 6\left(k + (k-1) + \cdots + 1\right) + k \cdot \frac{r}{2} - 3k + T(r) \\
& = 3k^2 + \frac{1}{2}kr + T(r) \\
& = \frac{(12k+r)^2}{48} - \frac{r^2}{48} + T(r)
\end{aligned}
$$

One checks that $T(r) - \frac{r^2}{48}$ is a fraction strictly between $-\frac{1}{2}$ and $\frac{1}{2}$ for each of the six values of r. That is, for each even value N:
\

$$T(N) = \frac{N^2}{48} \pm \varepsilon$$

for some small fraction ε less than one half. Thus $T(N) = \left\langle \frac{N^2}{48} \right\rangle$ as claimed in the newsletter. For N odd we have $T(N) = T(N+3) = \left\langle \frac{(N+3)^2}{48} \right\rangle$.

CHALLENGE: Let $S(n)$ be the count of *scalene* integer triangles of perimeter *n*. What do you notice about the sequence of numbers produced?

FINAL COMMENT ON THE AREA AND PERIMTER OF POLYGONS:

Suppose a polygon of area A and perimeter P has sides of lengths a_1, a_2, $\ldots$, a_k. Consider the quantity $\frac{P}{2A} = \frac{a_1 + a_2 + \cdots + a_k}{2A}$.

Now area is computed as the product of two quantities of dimension "length" and if there is a natural means to use a sum of some of those side lengths in the computation of area, $a_{i_1} + \cdots + a_{i_n}$ say, then $h = \frac{2A}{a_{i_1} + \cdots + a_{i_n}}$ is a "meaningful" length in the geometry of the figure and its reciprocal $\frac{1}{h}$ appears in the expression $\frac{P}{2A}$.

For example, for a triangle with sides a, b and c, its area can be computed as $A = \frac{1}{2}(a+b+c)r$ where r is the inradius of the triangle and:

$$\frac{P}{2A} = \frac{1}{r}$$

Its area can also be computed as $A = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$ (where h_a, h_b and h_c are the altitudes of the triangle) and

$$\frac{P}{2A} = \frac{a}{2A} + \frac{b}{2A} + \frac{c}{2A} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

Does thinking along these lines add to or diminish the idea that something interesting is going on in the Curious Challenge presented earlier in this chapter?

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[TANTON]
Tanton, J., "Pit your wits against young minds!" *Mathematical Intelligencer*, **29** no. 3 (2007), 55-59.