PERMUTATIONS AND COMBINATIONS

HOW TO AVOID THEM AT ALL COSTS AND STILL ACTUALLY UNDERSTAND AND DO COUNTING PROBLEMS WITH EASE!

A BRIEF FOUR-STEP PROGRAM

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COMMENT: If I were lord of the mathematics teaching universe I would, on day one, eliminate the phrase "does order matter" from all texts and all minds!

ANOTHER COMMENT: The following is adapted from

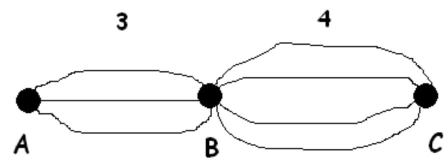
***THINKING MATHEMATICS*! Volume II**

available at the above website.

STEP 1: THE MULTIPLICATION PRINCIPLE

Let's start with a tiny puzzle:

There are three major highways from Adelaide to Brisbane, and four major highways from Brisbane to Canberra.



How many different routes can one take to travel from Adelaide to Canberra?

Exercise: Is the answer **7** (from $3+4$) or is the answer **12** (from 3×4)? Be very clear in your own mind as to whether one should add or should multiply these numbers.

Suppose there are also six major highways from Canberra to Darwin.

How many different routes are there from A to D?

Exercise: Explain why the answer is $3\times 4\times 6 = 72$.

I own five different shirts, four different pairs of trousers and two sets of shoes. How many different outfits could you see me in?

Answer: ???

There are ten possible movies I can see and ten possible snacks I can eat whilst at the movies. I am going to see a film tonight and I will eat a snack. How many choices do I have in all for a movie/snack combo?

Answer: ???

We have a general principle:

THE MULTIPLICATION PRINCIPLE
If there are a ways to complete one task and b ways to complete a second task, and the outcomes of the first task in no way affect the choices made for the second task, then the number of different ways to complete both tasks is $a \times b$.

This principle readily extends to the completion of more than one task.

Exercise: Explain the clause stated in the middle of the multiplication principle. What could happen if different outcomes from the first task affect choices available for the second task? [For example, suppose I will never wear my purple shirt with my green trousers? Would this ruin the multiplication principle?]

EXAMPLE: On a multiple choice quiz there are five questions, each with three choices for an answer:

1.	**A**	**B**	**C**
2.	**A**	**B**	**C**
3.	**A**	**B**	**C**
4.	**A**	**B**	**C**
5.	**A**	**B**	**C**

I decide to fill out my answers randomly. In how many different ways could I fill out the quiz?

Answer: This is a five-stage process:

<u>Stage 1</u>: Answer question one: 3 ways
<u>Stage 2</u>: Answer question two: 3 ways
<u>Stage 3</u>: Answer question three: 3 ways
<u>Stage 4</u>: Answer question four: 3 ways
<u>Stage 5</u>: Answer question five: 3 ways

By the multiplication principle there are $3 \times 3 \times 3 \times 3 \times 3 = 3^5$ ways to complete the quiz. □

STEP 2: REARRANGING LETTERS

My name is JIM. In how many ways can I rearrange the letters of my name?

Answer 1: Let's just list the ways.

JIM
JMI
MJI
MIJ
IJM
IMJ

There are six ways.

Answer 2: Use the multiplication principle: We have three slots to fill:

$$\_\_\_ \quad \_\_\_ \quad \_\_\_$$

The first task is to fill the first slot with a letter. There are 3 ways to complete this task.

The second task is to fill the second slot. There are 2 ways to complete this task. (Once the first slot is filled, there are only two choices of letters to use for the second slot.)

The third task is to fill the third slot. There is only 1 way to complete this task (once slots one and two are filled).

$$\underline{\ 3\ }\ \underline{\ 2\ }\ \underline{\ 1\ }$$

By the multiplication principle, there are thus $3 \times 2 \times 1 = 3!$ ways to complete this task. □

In how many ways can one arrange the letters HOUSE ?

In how many ways can one arrange the letters BOVINE ?

In how many ways can one arrange the letters FACETIOUS ?
(What is unusual about the vowels of this word? Find another word in the English language with this property!)

When playing with these problems the following definition seems to be in need:

Definition: The product of integers from 1 to N is called N *factorial* and is denoted $N!$.
These numbers grow very large very quickly:

$$1! = 1$$
$$2! = 2 \times 1 = 2$$
$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$
$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$
$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$
$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$
$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

What is the first factorial larger than a million? A billion?

ON A CALCULATOR … There is a Factorial feature hidden under the MATH button under the PROBABILITY menu.

Now let's take it up a notch:

How many ways are there to rearrange the letters BOB? Assume the Bs are indistinguishable?

Exercise: Answer this question! Do you think the only way to answer it is to list all the possibilities?

In how many ways can one arrange the letters HOUSES?

Exercise: Try answering this one. Does listing all the possibilities seem fun?

Here's a clever way to think of this problem …

If the Ss were distinguishable – written, say, as S_1 and S_2 - then the problem is easy to answer:

There are 6! ways to rearrange the letters $HOUS_1ES_2$.

The list of arrangements might begin:

$HOUS_1ES_2$
$HOUS_2ES_1$

$OHUS_1S_2E$
$OHUS_2S_1E$

S_1S_2UEOH
S_2S_1UEOH
⋮

But notice, if the Ss are no longer distinguishable, then pairs in this list of answers "collapse" to give the same arrangement. We must alter our answer by a factor of two and so the number of arrangements of the word HOUSES is:

$$\frac{6!}{2} = 360$$

How many ways are there to rearrange the letters of the word CHEESE?

Answer: If the three Es are distinct – written E_1, E_2, and E_3, say – then there are $6!$ ways to rearrange the letters $CHE_1E_2S\,E_3$. But the three Es can be rearranged $3!=6$ different ways within any one particular arrangement of letters. These six arrangements would be seen as the same if the Es were no longer distinct:

$$\begin{matrix} HE_1E_2SCE_3 & HE_3E_1SCE_2 & & \\ HE_1E_3SCE_2 & HE_3E_2SCE_1 & \rightarrow & HEESCE \\ HE_2E_1SCE_3 & HE_2E_3SCE_1 & & \end{matrix}$$

Thus we must divide our answer of $6!$ by $3!$ to account for the groupings of six that become identical. There are thus $\frac{6!}{3!}=\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1}=6\cdot5\cdot4=120$ ways to arrange the letters of CHEESE. □

Comment: The number of ways to rearrange the letters HOUSES is $\frac{6!}{2!}$. The "2" on the denominator is really 2!.

Explain why the number of ways to arrange the letters of the word CHEESES is $\frac{7!}{3!2!}$.

In how many ways can one arrange the letters CHEEEEESIEST?

QUESTION: Consider the word CHEESIESTESSNESS. Do you see that there are

$$\frac{16!}{5!6!}$$

ways to rearrange its letters? Evaluate this number.

Comment: It is actually better to write this answer as

$$\frac{16!}{1!1!5!6!1!1!1!}$$

$1!$ for the one letter C
$1!$ for the one letter H
$5!$ for the five letters E
$6!$ for the four letters S
$1!$ for the one letter I
$1!$ for the one letter T
$1!$ for the one letter N

This offers a "self check" : The numbers appearing on the bottom should sum to the number appearing on the top.

Question: Does $1!$ make sense?

How many ways can you rearrange the letters of your full name?

***** EXTRA *****

Evaluate the following expressions:

a) $\frac{800!}{799!}$ b) $\frac{15!}{13!2!}$ c) $\frac{87!}{89!}$

Simplify the following expressions as far as possible:

d) $\frac{N!}{N!}$ e) $\frac{N!}{(N-1)!}$ f) $\frac{n!}{(n-2)!}$

g) $\frac{1}{k+1}\cdot\frac{(k+2)!}{k!}$ h) $\frac{n!(n-2)!}{((n-1)!)^2}$

STEP 3: THE LABELING PRINCIPLE

How many ways can you rearrange the letters of the Swedish pop group ABBA?

Answer: $\frac{4!}{2!2!} = \frac{24}{4} = 6$.

How many ways can you rearrange the letters of AABBBBA?

Answer: $\frac{7!}{3!4!}$

How many ways can you rearrange the letters of AAABBBBCCCCCC?

Answer: $\frac{13!}{3!4!6!}$

Let's look at this third problem and phrase it in a different way:

Mean Mr. Muckins has a class of 13 students. He has decided to call three of the students A students, four of them B students, and six of them C students. In how many ways could he assign these labels?

Answer: Let's imagine all thirteen are in a line:

$$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$$

Here's one way he can assign labels:

$$\underline{A}\ \underline{C}\ \underline{B}\ \underline{B}\ \underline{B}\ \underline{A}\ \underline{C}\ \underline{C}\ \underline{C}\ \underline{C}\ \underline{A}\ \underline{C}\ \underline{B}$$

Here's another way:

$$\underline{B}\ \underline{A}\ \underline{C}\ \underline{C}\ \underline{B}\ \underline{C}\ \underline{C}\ \underline{C}\ \underline{B}\ \underline{A}\ \underline{C}\ \underline{B}\ \underline{A}$$

and so on.

We see that this labeling problem is just the same problem as rearranging letters. The answer must be $\frac{13!}{3!4!6!}$. $\square$

There are 10 people in an office and 4 are needed for a committee. How many ways?

Answer: Imagine the 10 people standing in a line. We need to give out labels. Four people will be called "ON" and six people will be called "LUCKY." Here is one way to assign those labels:

$$\underline{L}\ \underline{L}\ \underline{O}\ \underline{O}\ \underline{L}\ \underline{O}\ \underline{L}\ \underline{L}\ \underline{L}\ \underline{O}$$

We see that this is just a word arrangement problem. The answer is:

$$\frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210$$

□

In general, we have …

THE LABELING PRINCIPLE

Each of distinct N objects is to be given a label.

If a of them are to have label "1," b of them to have label "2," and so on, then the total number of ways to assign labels is:

$$\frac{N!}{a!b!\cdots z!}$$

PUTTING THE LABELING PRINCIPLE TO USE:

Three people from a group of twelve are needed for a committee. In how many different ways can a committee be formed?

Answer: The twelve folk are to be labeled as follows: 3 as "on the committee" and 9 as "lucky." The answer must be $\frac{12!}{3!\,9!} = 220$. □

COMMENT: Notice that we were sure to assign an appropriate label to each and every person (or object) in the problem.

Fifteen horses run a race. How many possibilities are there for first, second, and third place?

Answer: One horse will be labeled "first," one will be labeled "second," one "third," and twelve will be labeled "losers." The answer must be: $\frac{15!}{1!\,1!1!12!}$. □

A "feel good" running race has 20 participants. Three will be deemed equal "first place winners," five will be deemed "equal second place winners," and the rest will be deemed "equal third place winners." How many different outcomes can occur?

Answer: Easy! $\frac{20!}{3!\,5!12!}$. □

From an office of 20 people, two committees are needed. The first committee shall have 7 members, one of which shall be the chair and 1 the treasurer. The second committee shall have 8 members. This committee will have 3 co-chairs and 2 co-secretaries and 1 treasurer. In how many ways can this be done?

Answer: Keep track of the labels. Here they are:

1 person will be labeled "chair of first committee"
1 person will be labeled "treasure of first committee"
5 people will be labeled "ordinary members of first committee"
3 people will be labeled "co-chairs of second committee"
2 people will be labeled "co-secretaries of second committee"
1 person will be labeled "treasurer of second committee"
2 people will be labeled "ordinary members of the second committee"
5 people will be labeled "lucky," they are on neither committee.

The total number of possibilities is thus: $\frac{20!}{1!1!5!3!2!1!2!5!}$. Easy! □

Suppose 5 people are to be chosen from 12 and the order in which folk are chosen is not important. How many ways can this be done?

Answer:
5 people will be labeled "chosen" and 7 "not chosen. There are $\frac{12!}{5!7!}$ ways to accomplish this task. □

Suppose 5 people are to be chosen from 12 for a team and the order in which they are chosen is considered important. In how many ways can this be done?

Answer: We have:

1 person labeled "first"
1 person labeled "second"
1 person labeled "third"
1 person labeled "fourth"
1 person labeled "fifth"
7 people labeled "not chosen"

This can be done $\frac{12!}{1!1!1!1!1!7!}$ ways. □

STEP 4: MULTIPLICATION AND LABELING PRINCIPLES TOGETHER

There are 7 men and 6 women in an office. How many ways are there to make a committee of five if …

a) Gender is irrelevant?
b) The committee must be all male?
c) The committee must consist of 2 men and 3 women?

Answer: a) This is just a problem of assigning labels to 13 people: five are "ON" and eight are "OFF":

$$\frac{13!}{5!8!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 11 \cdot 9 = 1287 \text{ ways}$$

b) This is a problem just among the men: 7 people to be labeled:

$$\frac{7!}{5!2!} = 21 \text{ ways}$$

c) **THIS IS A TWO STAGE PROBLEM!**

Stage 1: Deal with the men

$$\frac{7!}{2!5!} = 21 \text{ ways}$$

Stage 2: Deal with the women

$$\frac{6!}{3!3!} = 20 \text{ ways}$$

By the multiplication principle, there are $21 \times 20 = 420$ ways to form the committee. □

There are:

20 Americans
10 Australians (which include Dr. T.)
10 Austrians

Fourteen are needed for a math team. How many ways if …

a) Nationality is irrelevant?
b) The team must be all American?
c) The team must have 5 Americans, 5 Australians, and 4 Austrians?
d) Nationality is irrelevant, but Dr. T. must be on the team?

Answer:

a) $\frac{40!}{14!26!}$ (Do you see why?)

b) $\frac{20!}{14!6!}$ (Do you see why?)

c) Deal with Americans: $\frac{20!}{5!15!}$

Deal with the Australians: $\frac{10!}{5!5!}$

Deal with the Austrians: $\frac{10!}{4!6!}$

By the multiplication principle there are a total of $\frac{20!}{5!15!}\times\frac{10!}{5!5!}\times\frac{10!}{4!6!}$ ways to make a team.

d) With Dr. T. on the problem is now to select 13 more team members from 39 people. There are $\frac{39!}{13!26!}$ ways to do this.

□

In how many ways can one arrange the letters AMERICA …

a) if there are no restrictions?
b) if the rearrangement must begin with M?
c) if the rearrangement must begin with M and end with C?
d) if the rearrangement must begin with A?

Answer: a) $\frac{7!}{2!}$ (Why?)

b) This is just a matter of arranging the letters AERICA in six slots:

M
___ ___ ___ ___ ___ ___ ___

There are $\frac{6!}{2!}$ ways.

c) This is a matter of arranging the letters AERIA in five slots:

M C
___ ___ ___ ___ ___ ___ ___

There are $\frac{5!}{2!}$ ways.

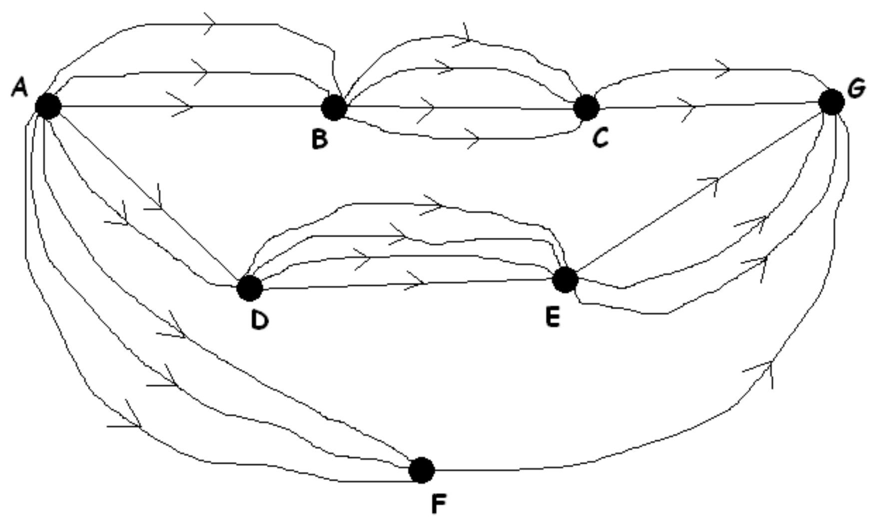
d) We need to arrange the letters MERICA in six slots:

A
___ ___ ___ ___ ___ ___ ___

There are $6!$ ways. □

STUDENT EXERCISES:

Question 1: How many different paths are there from A to G?



Question 2:

a) In how many different ways can one arrange five As and five Bs.
b) A coin is tossed 10 times. In how many different ways could exactly five heads appear?

Question 3: The word BOOKKEEPING is the only word in the English language with three consecutive double letters. In how many ways can one arrange the letters of this word?

Question 4: A multiple choice quiz has 10 questions each with 4 different possible answers. In how many ways can one fill out the quiz?

Question 5: In how many ways can one write down three vowels in order from left to right? How does the answer change if we insist that the vowels all be different?

Question 6: Ten people are up for election. In how many ways can one fill out a ballot for "president" and for "vice president"?

Question 7:

a) A mathematics department has 10 members. Four members are to be selected for a committee. In how many different ways can this be done?
b) A physics department has 10 members and a committee of four is needed. In that committee, one person is to be selected as "chair." In how many different ways can one form a committee of four with one chair?
c) An arts department has 10 members and a committee of four is needed. This committee requires two co-chairs. In how many different ways can one form a committee of four with two co-chairs?
d) An English department has 10 members and two committees are needed: One with four members with two co-chairs and one with three members and a single chair. In how many different ways can this be done?

Question 8: Three people from 10 will be asked to sit on a bench: one on the left end, one in the middle, and one on the right end. In how many different ways can this be done?

Question 9: Five pink marbles, two red marbles, and three rose marbles are to be arranged in a row. If marbles of the same colour are identical, in how many different ways can these marbles be arranged?

Question 10:

a) Hats are to be distributed to 20 people at a party. Five hats are red, five hats are blue, and 10 hats are purple. In how many different ways can this be done? (Assume the people are mingling and moving about.)
b) ***CHALLENGE:*** If the 20 people are clones and cannot be distinguished, in how many essentially different ways can these hats be distributed?

Question 11: In how many ways can one arrange the letters of NOODLEDOODLE if the arrangement must begin with an L and end with an E?

Question 12: A committee of five must be formed from five men and seven women.

a) How many committees can be formed if gender is irrelevant?
b) How many committees can be formed if there must be at exactly two women on the committee?
c) How many committees can be formed if one particular man must be on the committee and one particular woman must not be on the committee?
d) **CHALLENGE:** How many committees can be formed if one particular couple (one man and one woman) can't be on the committee together?

Question 13:

a) Twelve white dots lie in a row. Two are to be coloured red. In how many ways can this be done?
b) Consider the equation $10 = x + y + z$. How many solutions does it have if each variable is to be a positive integer or zero?

Question 14:

a) In how many ways can the letters ABCDEFGH be arranged?
a) In how many ways can the letters ABCDEFGH be arranged with letter G appearing somewhere to the left of letter D?
b) In how many ways can the letters ABCDEFGH be arranged with the letters F and H adjacent?