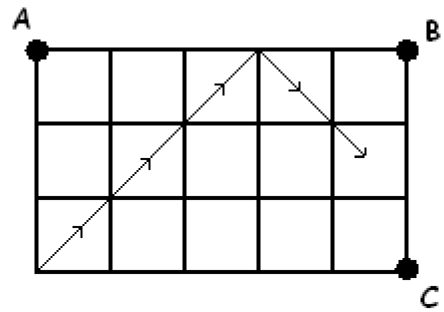
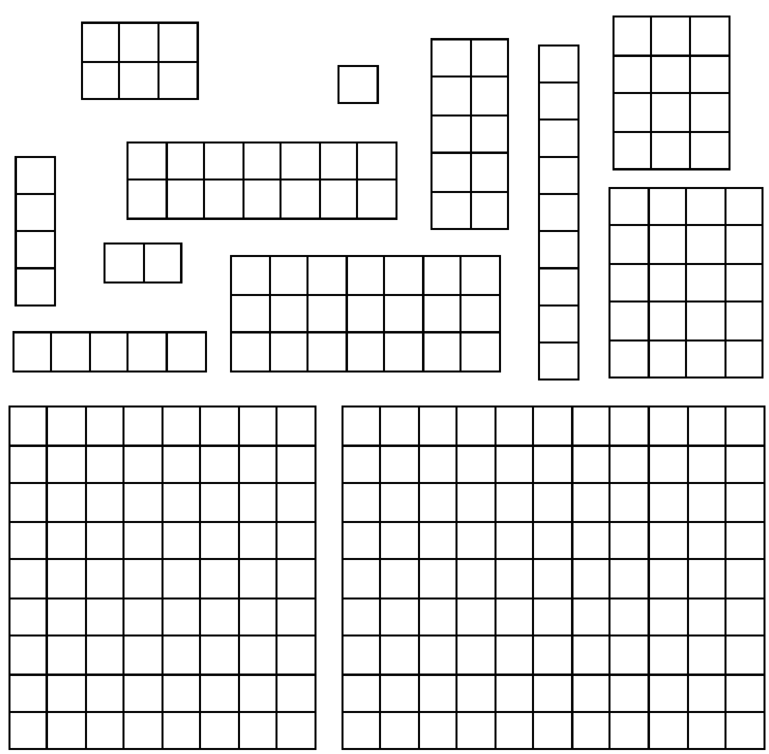
REFLECTION MATH!

A popular discovery activity in the middle school curriculum proceeds as follows:

A ball is shot from the bottom left corner of a 3×5 billiard table at a 45 degree angle. The ball traverses the diagonals of individual squares drawn on the table, bouncing off the sides of the table at equal angles. Into which pocket, A, B, or C, will the ball eventually fall?



Experiment with tables of the different dimensions above. What do you notice about those tables that have the ball fall into the top-left pocket A? Into the top-right pocket B? Into the bottom-right pocket C? Test your theories with more tables.



After some experimentation one notices that a ball bouncing in an $n \times m$ table with both *n* and *m* odd ends in pocket B; with *n* even and *m* odd in pocket A; and with *n* odd and *m* even, pocket C. Matters are a tad more obscure if both *n* and *m* are even. (You may have noticed that I avoided such tables among the previous diagrams.)

Often the investigation ends here, but there are some interesting questions about this activity that should be asked – and answered!

Question 1: Must every ball land in either pocket A, B, or C? Is it possible for a ball to return to start? Is it possible for a ball to enter an "infinite loop" and never fall into a pocket?

Question 2: Color the cells of each table black and red like a checkerboard. What do you notice about the path of the ball across the red cells? Across the black cells? Imagine (do not draw!) a 13×29 table. Use what you observe to explain why the ball can only possibly fall into pocket B. Explain the observations made at the top of this column.

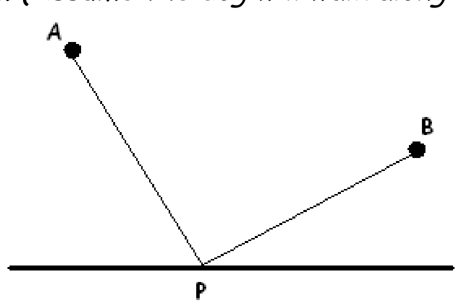
Question 3: Did you notice that the ball passes through each and every cell for each of the tables above? Not every billiard table has this property. Find a condition on the numbers *n* and *m* that ensures that this will be the case for an $n \times m$ table.

Question 4: Develop a theory that will predict into which pocket the ball will fall if both *n* and *m* are even.

ANOTHER TIDBIT: THE SAME PROBLEM TWICE

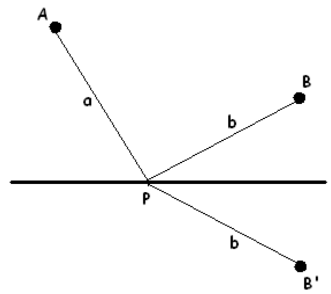
Answering the same question in more than one way can be incredibly illuminating. Behold!

A dog, starting at point A wishes to walk to point B via a path that first visits a wall. (Assume the dog will walk along two straight line segments to do this.)

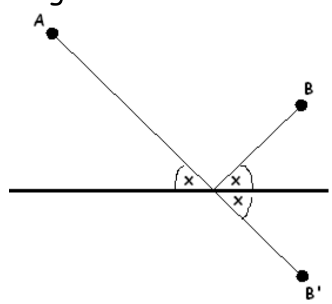


Locate the point P on the wall that gives the shortest possible path from A to P to B.

ANSWER ONE: Here's a sneaky trick. Let *B'* be the reflection of point *B* on the other side of the wall. Ignoring the issue that it is impossible for the dog to walk through the wall, notice that any path from A to a point P to B' is matched by a path of equal length from A to P to B, and vice versa.

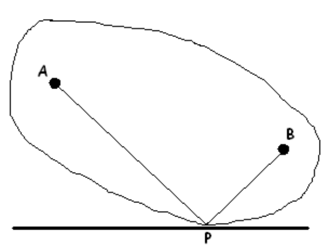


Clearly the shortest path from A to B' is the straight line path, making equal angles as shown. Consequently the shortest path from A to B via the wall is the one that "bounces off" the wall at equal angles.



ANSWER TWO: Recall that an ellipse with foci points A and B is the curve with the property that all points P on the curve give the same path length A to P to B.

Draw an ellipse about the points A and B. If the ellipse is too small it won't reach the wall and give an appropriate dog walking path. If it is too big, it can be replaced a smaller ellipse giving a shorter path from A to P to B. Thus the point P we seek is the location where the smallest ellipse possible just touches the wall. That is, the wall represents a tangent line to an ellipse and P is the point of contact.



In both answers the point P must be the same. We have thus proven the famous reflection property of an ellipse! **Any path from one focus of an ellipse to the other via a point on the ellipse makes equal angles to the (tangent line) to the ellipse.** Thus a ball thrown from one focus to any point on the wall of the ellipse will head directly through the second focus. Sound waves in the elliptical Mormon Tabernacle, UT, and the Whispering Gallery in the United States Capitol building, D.C., operate this way! (Notice that the sound waves will bounce back and forth infinitely often between the two foci!)

RESEARCH CORNER: THE OTHER REFLECTION PROPERTIES?
Devise other problems that can be solved in two different ways to prove the famous reflection properties of a parabola and a hyperbola.

THE FULL STORY

This charming billiard activity is simply super. It provides many avenues of exploration for both young and advanced students (and if one wants to explore billiard action on general polygonal-shaped tables, one can enter the world of current active research!).

The questions I pose in the newsletter are not usually examined at the school level. When given permission to think openly students of all ages will ask them!

Question 1: *Must every ball land in either pocket A, B, or C. Is it possible for a ball to return to start? Is it possible for a ball to enter an "infinite loop" and never fall into a pocket?*

Notice that the motion of the ball is "time reversible:" If the ball is currently traversing one particular cell (in a particular direction), then there is absolutely no doubt from which cell it just came. This observation is key.

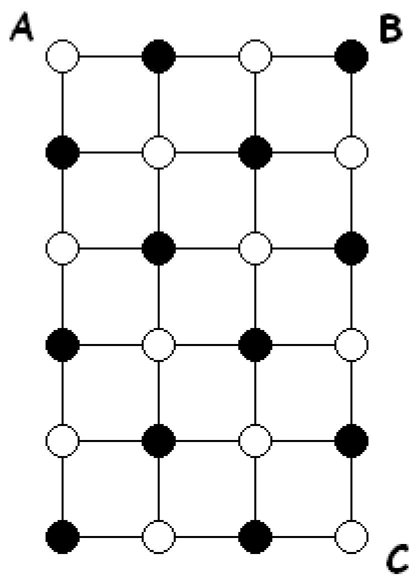
Could the ball enter the same square twice in the same direction? No! Each time the ball enters a "repeat square" it must have come from a previous repeat square. There is no first cell the ball can visit twice.

Could the ball enter the same square twice in the opposite directions? No! To do this the ball must have traversed a previous square twice in opposite directions, and so, again, there is no first cell the ball can traverse twice in this way. In particular, the ball will never return to the bottom left cell and return to start. (Could a ball traverse the same square but along different diagonals?)

As there are only finitely many cells in the grid and no cell can be visited twice along the same diagonal, the ball's path must terminate. This establishes that the ball is sure to fall into one of the pockets A, B or C.

Question 2: *Color the cells of each table black and red like a checkerboard. What do you notice about the path of the ball across the red cells? Across the black cells? Imagine (do not draw!) a* 13×29 *table. Use what you observe to explain why the ball can only possibly fall into pocket B. Explain the observations made in the paragraph at the top of this column.*

This question suggests a popular approach to analyzing billiard motion. (See XXXX, for instance.) One sees that the ball traverses red cells only along northwest diagonals and the black cells along southwest diagonals (or vice versa), and matters about the ball's behavior fall into place. But the analysis is slightly easier to color instead the grid points of the table alternately two colors in a checkerboard pattern. (We've chosen the colors black and white here.)



If we assume that the start corner is colored black, then it is clear that the ball will only ever visit black grid points. We also see:

> If n and m are both odd, then pockets A and C will be white and pocket B black. The ball must fall in pocket B.
>
> If n is even and m odd, then only pocket A is black. The ball must fall there.
>
> If n is odd and m even, only pocket C is black.

When n and m are both even, each corner cell is black. This case requires further analysis, which we shall conduct in a moment.

Question 3: *Did you notice that the ball passed through each and every cell in each of the tables above? Not every billiard table has this property. Find a condition on the numbers n and m that ensures that this will be the case for an $n \times m$ table.*

Let N be the number of cells visited by the bouncing ball. We certainly have $N \le nm$.

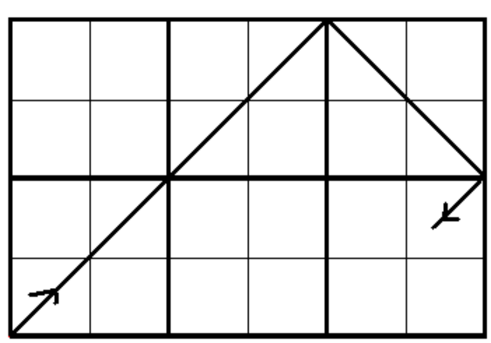
Now the ball moves left and right (as well as up and down) about the table to eventually fall into one of the pockets on the left side or the right side of the table. Looking at only the horizontal component of the motion of the ball, it is clear that N must be a multiple of m. By the same token, looking at the vertical component of the ball's motion, N must also be a multiple of n. If n and m are coprime (that is, share no common factors), then we have that N is a multiple of the product mn, and so $N \ge mn$ as well.

This shows that N is nm, the total number of cells in the grid if n and m share no non-trivial factors.

As we shall see next, if the numbers m and n are not coprime, if both are even, for instance, then some cells are sure to be missed.

Question 4: *Develop a theory that will predict into which pocket the ball will fall if both m and n are even.*

A 4x6 table, for example, is really just a 2x3 table in disguise:



and so the ball will fall into pocket A. A 10x50 table is really a 5x25 table in disguise (pocket B) and a 200x4000 table is really 1x20 table in disguise (pocket C).

In general, if n and m have greatest common factor d:

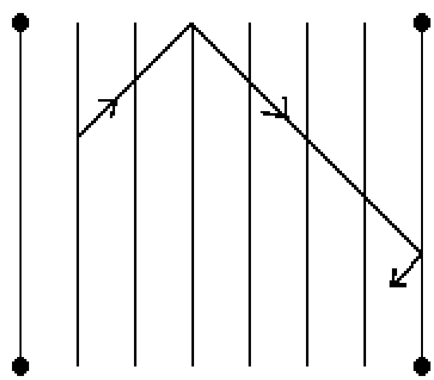
$$n = ad$$
$$m = bd$$

then an $n \times m$ table is really an $a \times b$ table in disguise with a and b coprime. The ball will traverse precisely abd cells (why this number?) and will land into pocket A, B, or C according to whether a and b are odd or even as dictated before.

MORE BILLIARDS

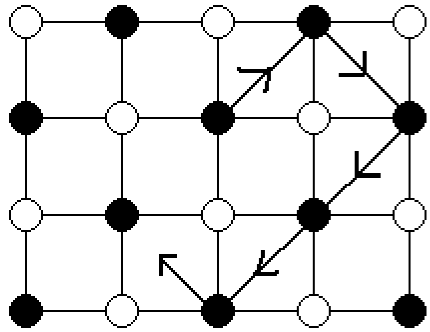
The mathematics behind the billiards game is surprisingly rich.

Suppose we start the billiard ball at <u>any</u> grid point of the table, setting it in motion along the diagonal of a cell. Assume that all four corners now represent pockets.



In playing with a 6×7 table, for example, one garners the impression that every starting point and starting direction yields a path ending in a pocket. Is the same true for a 14×25 table? A 6×8 table? What can one say about the behavior of billiard balls on general $n \times m$ grids? Perhaps play with some examples before reading on.

There are many subtleties in this exploration and a complete analysis of the situation requires some serious effort. We present here some partial results worth pondering upon just to get things going. Again assume we have again colored the grid points of an $n \times m$ table in a black and white checkerboard pattern, with the bottom-left corner black.



1. The ball only visits grid-points of the same color as its starting grid-point.

2. If a ball starts in a corner, it will terminate in a corner. Moreover, if $\gcd(n,m)=1$, the ball will fall into the only remaining pocket of the same color as its starting pocket and will pass through every cell of the table before doing so.

(We have essentially already established this.)

3. If n and m are both even, then any ball that starts at a white grid-point enters an infinite loop.

4. If $\gcd(n,m)=2$ and the ball starts at a white grid-point, then the ball enters an infinite loop that traverses each and every cell of the table.

5. If $\gcd(n,m)=2$, then any ball starting at a black grid-point will fall into a corner.

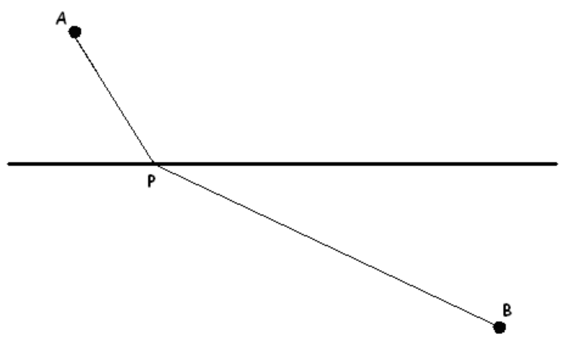
In playing with 4×4 and 4×8 tables, for example, one can find infinite loops that start at a white grid-point and do not pass through every cell of the table, and paths that start at black grid-points that do not fall into corners. The condition that $\gcd(n,m)=2$ is important here.

As this is an incomplete list of results, what then is the ultimate theorem that completely describes diagonal billiard-ball motion in rectangular tables?

RELECTION PROPERTIES OF CONICS

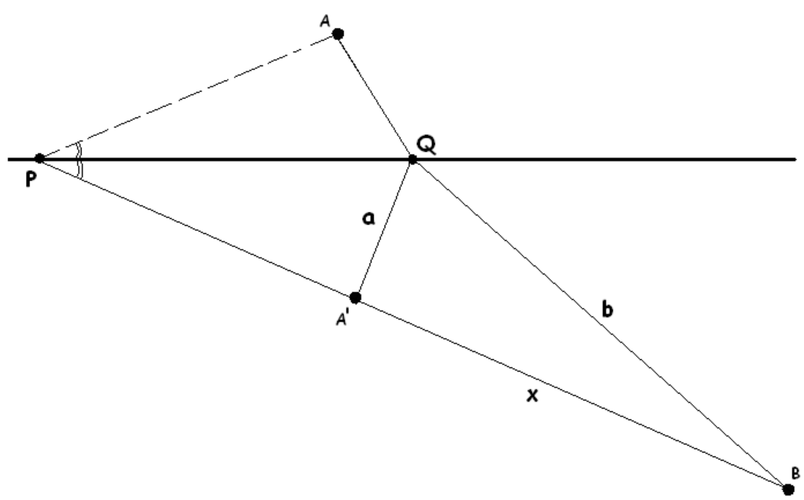
One can go remarkably far in solving optimization problems via reflections. For example, consider the following dual problem to the dog-walking problem:

Two points A and B lie on opposite sides of a line with point A slightly closer to the wall than point B.



Find the location of the point P on the line that *maximizes* *the* *difference* $|PB|-|PA|$.

ANSWER ONE: This time reflect the point *A* across the line to the point *A'*. Let *P* be the point on the line such that *P*, *A'* and *B* are collinear. For this point *P*, the value of $|PB|-|PA|$ is the length *x* shown.

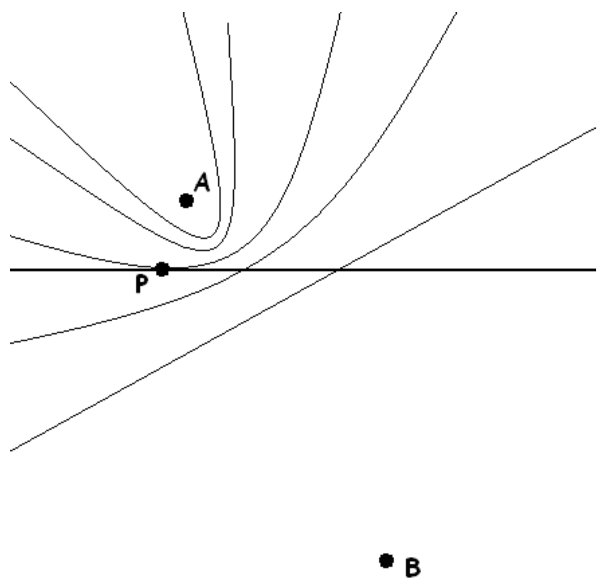


For any other point *Q* on the line, the analogous quantity is the difference of lengths $b-a$ shown. By the triangular inequality, $x+a>b$ and so $x>b-a$. The point P thus gives the maximal value for this quantity and is the point we seek.

Notice that this solution, like the answer to the first problem, relied on constructing the straight line between one of the points *A* and *B* and the reflection of the other. It too produces congruent angles.

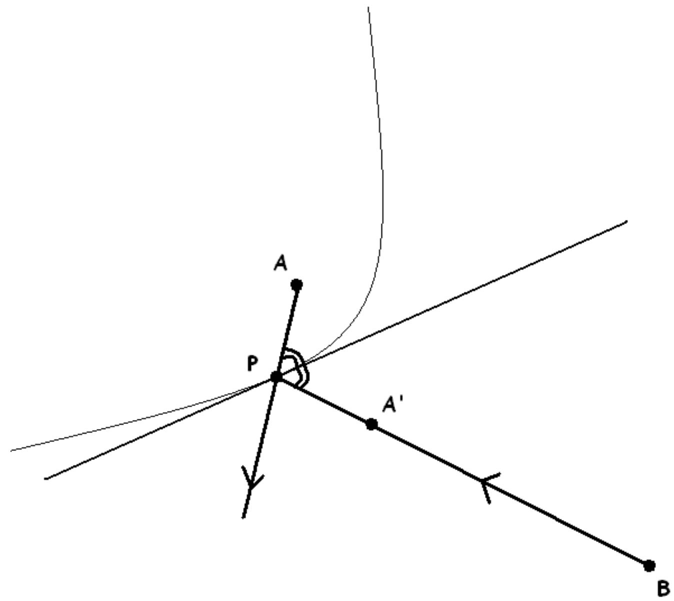
ANSWER TWO: Recall that a hyperbola with foci *A* and *B* is the locus of points *P* with the property that $|PB|-|PA|$ adopts a fixed value. (The largest this value can be is $|AB|$, the smallest is $-|AB|$. The cases with $|PB|-|PA|$ positive or negative represent the two different branches of the hyperbola. The case $|PB|-|PA|=0$ corresponds to the degenerate example of single straight line, the perpendicular bisector of $\overline{AB}$.)

Consider the locus of points P satisfying $|PB|-|PA|=k$ for a given value k with $0 \le k \le |AB|$. Each is a branch of a hyperbola that "wraps around" A, with $k=0$ yielding the straight perpendicular bisector. As k increases, we can find the "last" hyperbolic branch to touch the line. It gives the highest possible value of $|PB|-|PA|$ for points on the line, and this branch is tangent to the line.



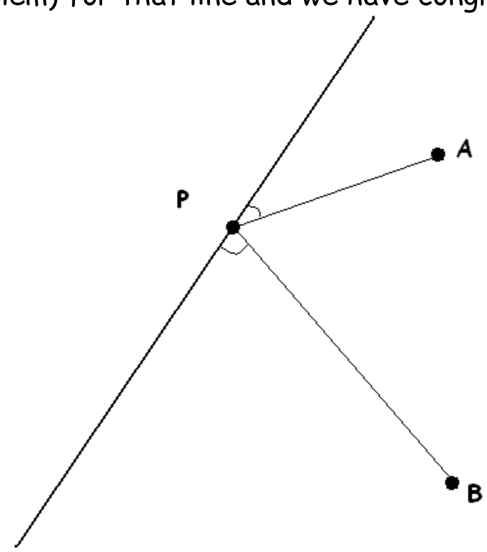
But we know that this point of contact *P* is the point obtained by extending the line segment $\overline{A'B}$ and that, for this point, congruent angles appear. We have established the reflection property of the hyperbola:

A ray of light directed from one focus of a hyperbola reflects off the hyperbola directly away from the second focus.



We can go further. Let A and B be two points in the plane. Then there is an infinite family of ellipses with A and B as foci and an infinite family of hyperbolae, also with A and B as foci.

Let P be an arbitrary point in the plane and consider the tangent line to an ellipse with foci A and B that passes through P. This point solves the minimization problem (the dog-walking problem) for that line and we have congruent angles.



Now draw a perpendicular line through *P* to see that the same diagram solves the maximization problem that defines the reflection property of a hyperbola!

This establishes that our two infinite families of ellipse and hyperbolae sharing the same foci are orthogonal: whenever one of each curve intersect at a point P their tangent lines at that point are perpendicular.

Now deduce that the infinite families of ellipses and hyperbolae are orthogonal. (That is, deduce that at each point in the plane, the ellipse and the hyperbola that each pass through that point intersect at that point at ninety-degree angles.)

We are left with one question:

> *Is there a minimization or maximization problem that identifies the reflection property of a parabola?*

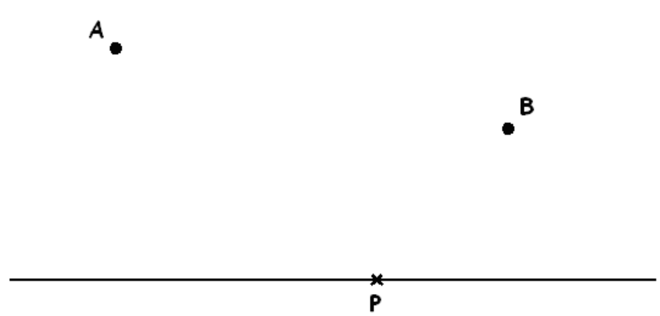
(For the study of ellipses we considered two points A and B on the same side of a line, and for hyperbolae points A and B on opposite sides of a line. What is the "intermediate" situation?)

Comment: The reflection properties of conics can also be explored/discovered through some activities in folding paper. See Appendix II of *THINKING MATHEMATICS! Volume 4*.

OTHER REFLECTION CHALLENGES:

The dog-walking challenge can be posed another way:

> *Abigail and Beatrice stand at positions A and B, respectively, in a squash court. Abigail wishes to hit a ball against the wall shown so that it bounces off and heads directly towards Beatrice. Describe the exact location of a point P along the wall towards which Abigail should aim to accomplish this feat. (Assume that Abigail and Beatrice live in an ideal world in which air resistance, friction, and inelasticity have absolutely no effect on the motion of bouncing balls.)*

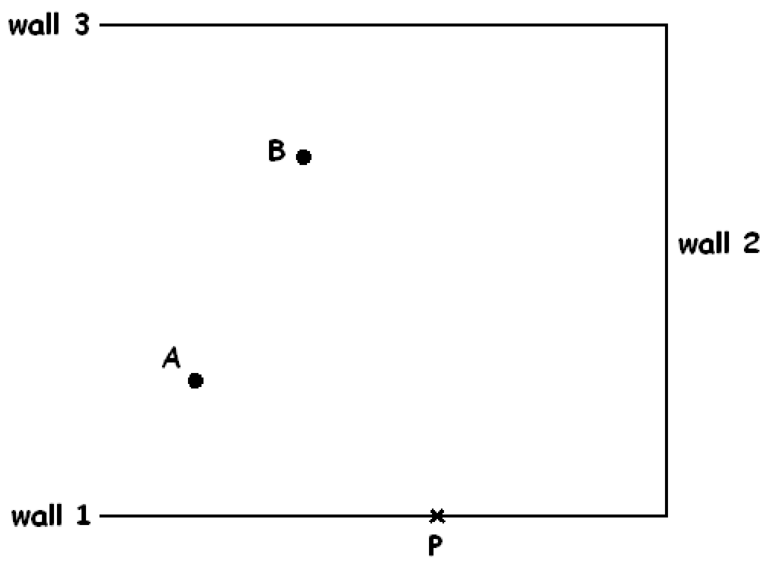


The point P that offers a path with angle of incidence matching angle of reflection does the trick. And, as we have seen, Abigail can find this special point by imagining that the wall is a mirror and simply aiming for Beatrice's reflection.

Question: Explain why, in ideal circumstances, a ball or light ray bouncing from a (mirrored) wall does so with equal angles of reflection.

Young students typically delight in thinking about a more complicated version of the previous challenge:

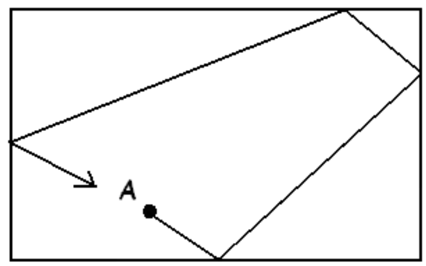
> *Abigail and Beatrice again stand at positions A and B, respectively, in a squash court, but this time Abigail wishes to make use of all three walls of the court. She hopes to hit a ball against wall 1 so that it then bounces off wall 2 to hit wall 3 and then head directly toward Beatrice. Describe the exact location of a point P along wall 1 towards which Abigail should aim to accomplish this amazing feat.*



Abigail solves this by aiming for the reflection of the reflection of the reflection of Beatrice.

This problem has some rich variations:

CHALLENGE 1: *Given any point in the interior of a rectangular billiards table describe how to hit the ball so that it is sure to return to start.*

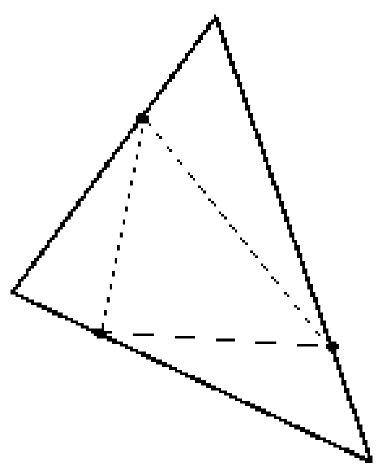


Under what conditions, if the ball were allowed to stay in motion, does the ball retrace precisely the same quadrilateral path over and over again in an infinite loop?

CHALLENGE 2: *Suppose we are given an acute triangle.*

a) *Show that there is an inscribed triangle with one vertex on each side of the original triangle of shortest perimeter.*

This problem was first posed, and solved, by Italian priest and scholar Giovanni Fagnano (1715-1797). It is today known as Fagnano's Problem. The internet offers plenty of information about it.



b) *Show that this inscribed triangle is the path of a ball bouncing inside the triangle caught in an "infinite loop."*

c) *Does an obtuse triangle also have an inscribed triangle of shortest perimeter?*