ON THE SQUARE ROOT OF TWO and THEON'S LADDER

Let's start with some puzzles and teasers. Full explanations will follow

PUZZLER: Irrational Numbers
Everyone "knows" that $\sqrt{2}$ is irrational.

a) Prove it! That is, prove there is something mathematically wrong in writing $\sqrt{2} = a/b$ for some integers a and b.
b) Prove that $\sqrt{3}$ is irrational.
c) Prove that $\sqrt{6}$ is irrational.
d) Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

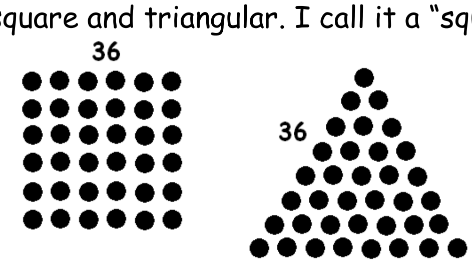
That $\sqrt{2}$ can be written as $2^{1/2}$ shows that it is possible to raise a rational number to a rational power to obtain an irrational result.

e) Is it possible to raise an irrational number to an irrational power to obtain a rational result?

SQUANGULAR NUMBERS:
The square numbers begin 1, 4, 9, 16, 25, …. One can arrange a square number of pebbles into a square array.

The triangle numbers begin: 1, 3, 6, 10, 15, …. One can arrange a triangle numbers of pebbles into a triangular array.

The number 36 is both square and triangular. I call it a "squangular number."



We have that 1 is the first squangular number, and 36 is the second. What's the third squangular number? How many squangular numbers are there?

Read on!

THEON'S LADDER: Theon of Smyrna (ca. 140 C.E.) knew that if a/b is a fraction that approximates $\sqrt{2} = 1.414...$, then $\dfrac{a+2b}{a+b}$ is a better approximation.

For example, $\frac{3}{2} = 1.5$ is close to $\sqrt{2}$, but $\frac{3+2\times 2}{3+2} = \frac{7}{5} = 1.4$ is closer still.

Starting with the initial fraction $1/1$ and iterating Theon's method gives a table of values today known as Theon's Ladder:

a	**1**	**3**	**7**	**17**	**41**	**99**	**239**	**577**	**1393**	...
b	**1**	**2**	**5**	**12**	**29**	**70**	**169**	**408**	**985**	...

[Notice: $1393/985 = 1.4142132....$]

Many wonderful patterns lie within this table!

1. Square the numbers in each row. We see that the top row is alternately one more and one less than double the bottom.

a^2	**1**	**9**	**49**	**289**	**1681**	**9801**	...
b^2	**1**	**4**	**25**	**144**	**841**	**4900**	...

[Algebra proves that this is so. If $a^2 = 2b^2 \pm 1$ then it is easy to check that $(a+2b)^2 = 2(a+b)^2 \mp 1$.]

This means that as we move along the table the quantity $\left(\frac{a}{b}\right)^2$ is adopting values closer and closer to the 2. Thus Theon's Ladder does indeed give better and better approximations to $\sqrt{2}$.

CHALLENGE: Establish that if a/b on the ladder differs from $\sqrt{2}$ by an amount E, then the next term $(a+2b)/(a+b)$ differs by an amount less than $\mathrm{E}/4$.

2. Look at every odd term of the ladder:

$\frac{1}{1}=\frac{0+1}{1}$ and $0^2+1^2=1^2$

$\frac{7}{5}=\frac{3+4}{5}$ and $3^2+4^2=5^2$

$\frac{41}{29}=\frac{20+21}{29}$ and $20^2+21^2=29^2$

$\frac{239}{169}=\frac{119+120}{169}$ and $119^2+120^2=169^2$

The odd terms of the ladder encode Pythagorean triple whose legs differ by one!

CHALLENGE: Prove that this ladder encodes <u>all</u> such Pythagorean triples.

3. Look at the even terms of the ladder:

$\frac{3}{2}=\frac{2(1)+1}{2(1)}$

$\frac{17}{12}=\frac{2(8)+1}{2(6)}$

$\frac{99}{70}=\frac{2(49)+1}{2(35)}$

$\frac{577}{408}=\frac{2(288)+1}{2(204)}$

If S_N and T_N denote, respectively, the Nth square and triangular numbers, then note:

$T_1=1$ and $S_1=1$
$T_8=36$ and $S_6=36$
$T_{49}=1225$ and $S_{35}=1225$
$T_{288}=41616$ and $S_{204}=41616$

The even terms encode squangular numbers!

CHALLENGE: Prove that ladder encodes <u>all</u> squangular numbers. (We see that there are infinitely many squangular numbers.)

3. The ladder encodes squangular numbers another way: Multiply together the entries in the top and bottom rows:

$1\times 1=1$ and $S_1=1$ is squangular.
$3\times 2=6$ and $S_6=36$ is squangular.
$7\times 5=35$ and $S_{35}=1225$ is squangular.
$17\times 12=204$ and $S_{204}=41616$ is squangular. And so on.

CHALLENGE: Establish this too!

COMMENT: The Swiss Mathematician Leonhard Euler (1707-1783) went a step further and proved that the Nth squangular number is given by the formula:

$$\left(\frac{\left(3+\sqrt{8}\right)^{N}-\left(3-\sqrt{8}\right)^{N}}{2\sqrt{8}}\right)^{2}$$

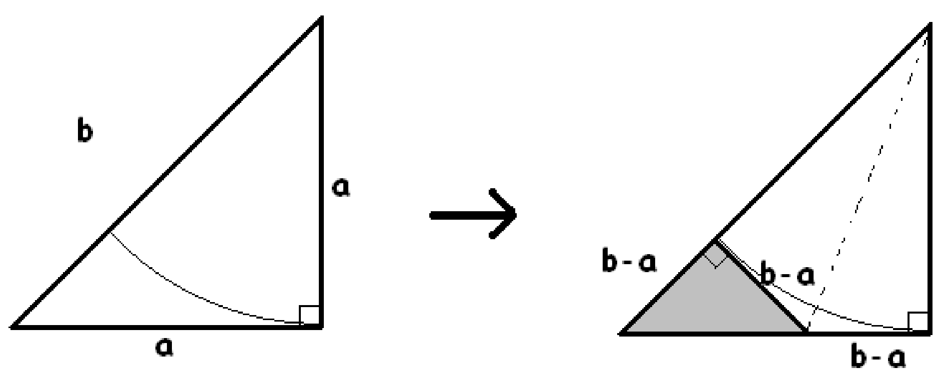
FURTHER CHALLENGES:
a) Discover other astounding patters in Theon's Ladder.
b) Start the ladder with a fraction different from $1/1$. What can you discover?
c) Create a ladder that approximates $\sqrt{3}$ or $\sqrt{5}$ or $\sqrt{6}$. Patterns? Are there analogs to the squangular numbers? (What about a ladder for $\sqrt{4}$?)
d) Is there a number that is simultaneously triangular, square and pentagonal?

THEON'S LADDER: FULL COMMENTARY

Indeed, everyone "knows" that the square root of two is irrational and school children are often given the impression that it is an obvious result. This, of course, is far from the case! Proving that the diagonal of a square is incommensurate with its side length takes some doing.

Theodorus of Cyrene (ca. 465-398 B.C.E.) developed a geometric argument that went as follows:

> *If the side of a square and its diagonal were commensurate, then there would be a unit of measure so that each has integral length. Half the square would be an isosceles right triangle with integer side lengths. Swing one leg along a circular arc as shown (or equivalently, in the modern world, fold a paper cut-out of the triangle along the dotted line). This produces a smaller right isosceles triangle again with integral side lengths. (Make use of similar triangles to see this or, observe closely the side-lengths if one folding paper.) We can repeat this process on the smaller and smaller right isosceles triangles that we produce until we obtain a right isosceles triangle with integer side-lengths smaller than the unit of measure! This absurdity shows that our starting assumption of commensurability must be wrong.*



Thedorus went on and developed geometric arguments to establish the irrationality of $\sqrt{n}$ for $n = 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15$ and 17.

Today many choose to establish the irrationality of $\sqrt{2}$ via a parity argument which proceeds as follows:

Suppose $\sqrt{2}=\frac{a}{b}$ *with* a *and* b *positive integers sharing no common factors. (That is, assume we have written* $\sqrt{2}$ *as a fraction in reduced form.) Then we must have* $2b^2=a^2$ *establishing that* a^2*, and hence* a*, is even. But then* $2b^2$ *is a square of an even number and so is a multiple of four, yielding that* b^2*, and hence* b *is also even. This contradictions the opening assumption.*

But it is not immediately clear how to generalize this argument to establish the irrationality of $\sqrt{3}$ and $\sqrt{6}$, yet alone other types of roots (the seventeenth root of 92, for instance).

A very effective means of establishing the irrationality of such numbers is to make use of the Fundamental Theorem of Arithmetic:

Two equal numbers must have identical prime factorizations (up to the order of the primes).

It is impossible, for instance, for $7\times7\times7\times11\times11\times13\times41\times41\times199$ to equal $37\times37\times37\times53\times53\times53\times53\times101\times101$. (The fundamental theorem was proved by Euclid, ca. 300 B.C.E. and is discussed in THINKING MATHEMATICS! Volume 1.) The theorem leads to the following useful result:

Lemma: *Suppose* a *and* b *are two positive integers sharing no common prime factors. If* $\frac{a}{b}$ *happens to be an integer, then* b *is* 1.

Proof: If $\frac{a}{b}=n$ for some integer n, then a and bn are two equal numbers possessing different primes in their factorizations, unless $b=1$ (and so $n=a$).

□

We can now classify which roots are rational:

Theorem: $\sqrt[n]{M}$ *is rational if, and only if,* M *is the* n *th power of an integer.*

Proof: If $\sqrt[n]{M}=\frac{a}{b}$ for some fraction in reduced form, then $\frac{a^n}{b^n}$ is an integer forcing b^n, and hence b, to be 1. □

Thus $\sqrt{3}$, $\sqrt{6}$ and $\sqrt[17]{92}$ are indeed irrational.

The sum of two irrationals can be rational, ($\sqrt{2}+\left(5-\sqrt{2}\right)$, for example, is rational) and it can be irrational: the sum $\sqrt{2}+\sqrt{3}$ is such an example. (If $\sqrt{2}+\sqrt{3}=\frac{c}{d}$, then $2+2\sqrt{6}+3=\frac{c^2}{d^2}$ showing that $\sqrt{6}=\frac{c^2-5d^2}{2d^2}$ is rational. This is not the case.)

CHALLENGE: Prove, using elementary techniques, that $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is irrational.

Similarly, multiplying two irrational numbers can lead to both rational and irrational results. Raising irrational numbers to irrational powers, however, is curious and tricky to analyze. Can an irrational number raised to an irrational power ever yield a rational result? The answer is … YES!

> *Consider $\sqrt{2}^{\sqrt{2}}$. If this happens to be rational, then we have an example of what we seek. If, on the other hand, $\sqrt{2}^{\sqrt{2}}$ is irrational, then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is an example of what we seek! (It is an irrational number raised to an irrational power and it equals $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}=\sqrt{2}^{2}=2$, a rational!) Either way, an example of what we desire exists. (We simply don't know whether the specimen is $\sqrt{2}^{\sqrt{2}}$ or $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$!!)*

CHALLENGE: Is there an example of an irrational number raised to an irrational power that is irrational?

COMMENT: The Gelfond-Schneider Theorem, easy to find on the internet, has something to say about this.

THEON'S LADDER

Theon's Ladder is a wonderfully rich and accessible topic for mathematical investigation. Students of all ages have explored the many properties of the ladder with me and I am forever astounded by the magic of the sequence and the myriad of directions one can go with it. Young students are particularly adept at discovering new and interesting patterns.

Here I outline the key features of the ladder, as suggested in the newsletter. The mathematics "behind the scenes" is Pell's equation, which, in and of itself, offers many rich avenues for deep exploration.

The first thing to note is that Theon's algorithm:

$$\frac{a}{b} \to \frac{a+2b}{a+b}$$

produces from a fraction in reduced form another fraction in reduced form. (If $a+2b$ and $a+b$ share a common factor d, then so do $(a+2b)-(a+b)=b$ and $2(a+b)-(a+2b)=a$. Thus if $\gcd(a,b)=1$, we have $d=1$.)

Theon's Ladder starts with the fraction $\frac{1}{1}$ and produces the sequence of terms,

a	1	3	7	17	41	99	239	577	1393	…
b	1	2	5	12	29	70	169	408	985	…

Let a_n and b_n denote the nth term of each row. Then we have:

$$\boxed{\begin{aligned} a_{n+1} &= a_n + 2b_n \\ b_{n+1} &= a_n + b_n \end{aligned}}$$

with $a_1 = b_1 = 1$. These show that the terms of $\{a_n\}$ are forever odd, and the terms of $\{b_n\}$ alternate in parity. We also see that both sequences are strictly increasing.

Notice that

$$a_{n+1}^2 - 2b_{n+1}^2 = (a_n + 2b_n)^2 - 2(a_n + b_n)^2 = -(a_n^2 - 2b_n^2)$$

From $a_1^2 - 2b_1^2 = -1$ we see:

$$\boxed{a_n^{\;2} - 2b_n^{\;2} = (-1)^n}$$

Thus $\left(\frac{a_n}{b_n}\right)^2 = 2 + \frac{(-1)^n}{b_n^{\;2}}$ and since $b_n \to \infty$ as $n \to \infty$ we do indeed have:

$$\boxed{\frac{a_n}{b_n} \to \sqrt{2} \text{ as } n \to \infty}$$

In the newsletter I claimed that the "error" in approximation decreases by a factor of at least four from term to term. To see this, note that:

$$\begin{aligned}
\left|\frac{a_{n+1}}{b_{n+1}} - \sqrt{2}\right| &= \left|\frac{a_n + 2b_n}{a_n + b_n} - \sqrt{2}\right| \\
&= \left(\sqrt{2}-1\right) \cdot \frac{1}{\frac{a_n}{b_n}+1} \cdot \left|\frac{a_n}{b_n} - \sqrt{2}\right| \\
&< \left(\sqrt{2}-1\right) \cdot \frac{1}{1+1} \cdot \left|\frac{a_n}{b_n} - \sqrt{2}\right| \\
&< \frac{1}{4} \cdot \left|\frac{a_n}{b_n} - \sqrt{2}\right|
\end{aligned}$$

(This is making use of the fact that $\frac{a_n}{b_n} \geq 1$.)

Connections to Pythagorean Triples and Squangular Numbers

We have the relation:

$$a_n^{\ 2} - 2b_n^{\ 2} = (-1)^n$$

This shows that the odd terms of the ladder satisfy

$$x^2 - 2y^2 = -1$$

(with x guaranteed to be odd), which can be rewritten $\left(\frac{x-1}{2}\right)^2 + \left(\frac{x+1}{2}\right)^2 = y^2$.

Thus for n odd, this reads $\left(\frac{a_n - 1}{2}\right)^2 + \left(\frac{a_n + 1}{2}\right)^2 = \left(b_n\right)^2$ giving a Pythagorean triple with legs differing by one.

The even terms of the ladder satisfy:

$$x^2 - 2y^2 = 1$$

(with x odd and y even), which can be rewritten $\frac{1}{2} \cdot \frac{x-1}{2} \cdot \frac{x+1}{2} = \left(\frac{y}{2}\right)^2$. If T_n denotes the nth triangular number ($T_n = 1 + 2 + \cdots + n = \frac{n\left(n+1\right)}{2}$) and S_n the nth square number ($S_n = n^2$) , then this equation reads $T_{\frac{x-1}{2}} = S_{\frac{y}{2}}$. This shows that, for n even, $\frac{a_n - 1}{2}$ and $\frac{b_n}{2}$ do indeed correspond to the indices of squangular numbers:

$$S_{\frac{b_n}{2}} = T_{\frac{a_n - 1}{2}} \quad (n \text{ even}).$$

The challenge now is to prove that every pair of positive integer solutions to $x^2 - 2y^2 = \pm 1$ does appear in Theon's Ladder (that is, that every Pythagorean triple with legs differing by 1 and every squangular number does appear in the way indicated above). Here goes:

Suppose (x, y) is a pair of positive integers satisfying $x^2 - 2y^2 = \pm 1$. Note that x and y cannot share a common factor different from 1 (then that factor would be a factor of ± 1 as well), thus the fraction $\frac{x}{y}$ is in reduced form. We need to show that $\frac{x}{y}$ appears in Theon's Ladder.

If y happens to be 1, then it is easy to see that $x^2 - 2y^2 = \pm 1$ gives x equal to 1 as well and we have the term $\frac{1}{1}$ in the ladder.

If y is not 1, then we have some work to do! Let's explore this case.

Just to get a feel for what we are looking for, if $\frac{x}{y}$ does appear in the ladder, what would be the term before it? One can check that following Theon's algorithm $\frac{a}{b} \to \frac{a+2b}{a+b}$ we come to the quantity $\frac{2y-x}{x-y}$, but we need to check that it is a meaningful fraction for the ladder. That is, we need to check that both the numerator and denominator are positive integers and that the fraction is in reduced form.

From $x^2 = 2y^2 \pm 1 = y^2 + \left(y^2 \pm 1\right)$ (with $y > 1$) we see that $x > y$. We also see that $x < 2y$ (square both sides). And it is straightforward to check that $2x - y$ and $x - y$ share no common factors if x and y do not. We're good!

Also note that from $x^2 - 2y^2 = \pm 1$ it follows that $(2y - x)^2 - 2(x - y)^2 = \mp 1$.

So what have we so far? If (x, y) is a solution to $x^2 - 2y^2 = \pm 1$, then $(2y - x, x - y)$ is another pair of positive integers satisfying the analogous equation and it is a solution with second term smaller than y. Also, if $\frac{x}{y}$ happened to be in Theon's ladder, then $\frac{2y-x}{x-y}$ would be the reduced fraction just before it in the ladder.

We're basically done!

We've established that from one solution $\frac{x}{y}$ we can obtain another solution with smaller denominator, and repeating this process we can obtain yet another solution with even smaller denominator. And we can keep repeating this. As we are staying in the realm of positive integers, we must eventually obtain a solution with denominator 1, which, we have shown must be the term $\frac{1}{1}$ in Theon's Ladder. As all the solutions we are creating come from applying the reverse of Theon's algorithm starting at $\frac{x}{y}$ to arrive at $\frac{1}{1}$, we have that $\frac{x}{y}$ arises from apply Theon's algorithm forward starting from $\frac{1}{1}$. Thus $\frac{x}{y}$ does indeed appear in the ladder, just as we hoped to show!

Going Further

Let's play with the recursion relations:

$$a_{n+1} = a_n + 2b_n$$
$$b_{n+1} = a_n + b_n$$

We have

$$a_{n+1} = a_n + b_n + b_n = b_{n+1} + b_n$$

and

$$b_{n+1} = a_n + b_n = b_n + b_{n-1} + b_n = 2b_n + b_{n-1}$$

that is:

$$\boxed{b_{n+1} = 2b_n + b_{n-1}}$$

with $b_1 = 1$ and $b_2 = 2$. It is convenient to set $b_0 = 0$.

What sequences satisfy this recursion relation?

It is a standard technique to examine whether or not a geometric sequence $1, x, x^2, x^3, x^4, \ldots$ could satisfy a given recursion relation. In our case we are looking for a value of x that satisfies:

$$x^{n+2} = 2x^{n+1} + x^n$$

that is, one that satisfies $x^2 = 2x + 1$. This has two solutions: $x_1 = 1+\sqrt{2}$ and $x_2 = 1-\sqrt{2}$.

Thus we have two geometric sequences that fit the recurrence relation:

$$1, x_1, x_1^2, x_1^3, x_1^4, \ldots$$

and

$$1, x_2, x_2^2, x_2^3, x_2^4, \ldots$$

It is straightforward to check that any linear combination of these two sequences would satisfy the same relation:

$$\alpha + \beta,\ \alpha x_1 + \beta x_2,\ \alpha x_1^2 + \beta x_2^2,\ \alpha x_1^3 + \beta x_2^3,\ \alpha x_1^4 + \beta x_4^2, \ldots$$

(We have: $ax_1^{n+2} + \beta x_2^{n+2} = 2\left(ax_1^{n+1} + \beta x_2^{n+1}\right) + \left(ax_1^{n} + \beta x_2^{n}\right)$.)

Let's choose values for α and β that match the initial values $b_0 = 0$ and $b_1 = 1$. We see that $\alpha = \frac{1}{2\sqrt{2}}$ and $\beta = -\frac{1}{2\sqrt{2}}$ do the trick. Now, if two sequences satisfy the same recursion relation and have the same initial conditions - as we have set up here - they must be the same sequence! We have thus established:

$$\boxed{b_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}}$$

Algebra gives:

$$\boxed{a_n = b_n + b_{n-1} = \frac{\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n}{2}}$$

CHALLENGE: Find a formula for $\frac{a_n}{b_n}$ and show, again, that $\frac{a_n}{b_n} \to \sqrt{2}$ as $n \to \infty$.

More algebra verifies that both $\frac{b_{2n}}{2}$ and $a_n b_n$ equal

$$\frac{\left(3+\sqrt{8}\right)^n - \left(3-\sqrt{8}\right)^n}{2\sqrt{8}}$$

This establishes the interesting relation $2a_n b_n = b_{2n}$ and it also establishes Euler's result: From $S_{\frac{b_{2n}}{2}} = T_{\frac{a_{2n}-1}{2}}$ it follows that the nth squangular number is:

$$\left(\frac{b_{2n}}{2}\right)^2 = \left(\frac{\left(3+\sqrt{8}\right)^n - \left(3-\sqrt{8}\right)^n}{2\sqrt{8}}\right)^2$$

Final Thoughts:

Starting Theon's Ladder with the fraction $\frac{5+12}{13} = \frac{17}{13}$ generates among its odd terms all Pythagorean triples with legs that differ by seven. What is hiding among the even terms?

Also, is it possible to extend Theon's Ladder infinitely far to the left? Are there meaningful geometric interpretations for the negative terms?

I do not know of a number that is simultaneously triangular, square and pentagonal, nor have any of my students found one. Might you find one?