PUSHING PYTHAGORAS

A triple of integers (a,b,c) is called a Pythagorean triple if $a^2 + b^2 = c^2$. For example, some classic triples are $(3,4,5)$, $(5,12,13)$, $(7,24,25)$. I am personally fond of $(20,21,29)$ and $(119,120,169)$.

Is there an easy way to find examples of such triples? Why yes! Just look at an ordinary multiplication table to find them!

x	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14
3	3	6	9	12	15	18	21
4	4	8	12	16	20	24	28
5	5	10	15	20	25	30	35
6	6	12	18	24	30	36	42
7	7	14	21	28	35	42	49

Choose any two numbers on the main diagonal (these are square numbers) and the two identical numbers to make a square of chosen figures. Sum the two square numbers, take the difference of the two square numbers, and sum the two identical numbers. You now have a Pythagorean triple! e.g.

$$a = 25 - 4 = 21$$
$$b = 10 + 10 = 20 \quad \Rightarrow \quad 20^2 + 21^2 = 29^2$$
$$c = 25 + 4 = 29$$

As another example, choose 36 and 1 to obtain:

$$a = 36 - 1 = 35$$
$$b = 6 + 6 = 12 \quad \Rightarrow \quad 12^2 + 35^2 = 37^2$$
$$c = 36 + 1 = 37$$

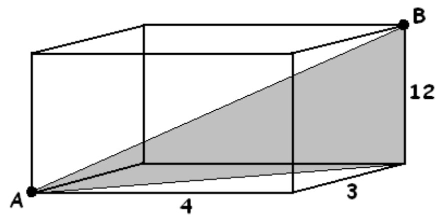
Question 1: Which two square numbers give the triple (3,4,5)? Which give the triples (5, 12, 13) and (7, 24, 25)?

Question 2: Why does this trick work?

Tough Challenge: This method fails to yield (9,12,15) but if the common factor of three is removed we do obtain (3,4,5). Show that every Pythagorean triple with no shared factors between the three terms does indeed appear via this method.

PYTHAGORAS MEETS THE THIRD DIMENSION

A classic problem in geometry is to work out the length of the longest diagonal in a rectangular box. For example, what is the distance between points A and B in this box?



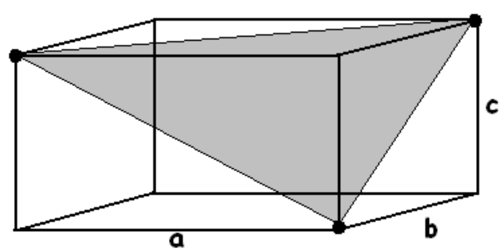
One application of the Pythagorean theorem shows that the length of the diagonal on the base of the figure is 5 units long. A second application of the theorem using the shaded triangle yields the length we seek as 13 units.

In general, precisely this argument shows that the length of the longest diagonal *d* in an $a \times b \times c$ rectangular box satisfies:

$$d^2 = a^2 + b^2 + c^2$$

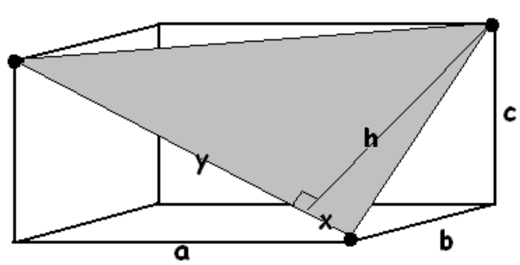
This is just one way to think of the Pythagorean theorem extended to the third dimension.

Consider the triangle sitting inside a rectangular box as shown. What is its area?



We can compute this with relative ease. (The algebra is a tad messy – but it is worth plowing through.). Start by removing the three right triangles about the top front corner of the box to give a clearer view.

Draw in an altitude for the triangle and label the lengths *x*, *y*, and *h* as shown:



Do you see that $x^2 + h^2 = b^2 + c^2$ and $y + x = \sqrt{a^2 + c^2}$? Do you also see that $y^2 + h^2 = a^2 + b^2$? Let's play with this third equation using the first two for support.

We have:

$$\left(\sqrt{a^2+c^2}-x\right)^2+h^2=a^2+b^2$$

So:

$$a^2+c^2-2x\sqrt{a^2+c^2}+x^2+h^2=a^2+b^2$$

yielding:

$$2x\sqrt{a^2+b^2}=x^2+h^2+c^2-b^2=2c^2$$

Thus:

$$x=\frac{c^2}{\sqrt{a^2+c^2}}$$

Now solve for h^2:

$$h^2=b^2+c^2-x^2=b^2+c^2-\frac{c^4}{a^2+c^2}$$
$$=\frac{a^2b^2+b^2c^2+a^2c^2}{a^2+c^2}$$

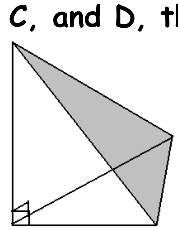
(Yick!)

Now we're all set to work out the area of the triangle: $A=\frac{1}{2}\times\sqrt{a^2+c^2}\times h$. ("Half base times height.") To avoid square roots let's compute instead A^2:

$$A^2=\frac{1}{4}(a^2+c^2)h^2=\frac{1}{4}(a^2b^2+b^2c^2+a^2c^2)$$
$$=\left(\frac{1}{2}ab\right)^2+\left(\frac{1}{2}bc\right)^2+\left(\frac{1}{2}ac\right)^2$$

Notice that each term squared is the area of one of the right triangles on the side of the rectangular box. So what have we got?

A triangle of area A sits across three mutually perpendicular right triangles. If these triangles have areas B, C, and D, then: $A^2=B^2+C^2+D^2$.

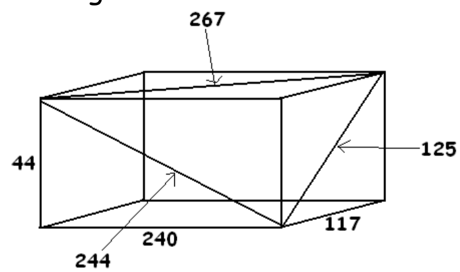


Exercise: A triangle crosses the *x*-, *y*- and *z*-axes at positions 3, 5 and 8. What is its area?

This alternative 3-D version of Pythagoras's theorem is not particularly well known!

RESEARCH CORNER:

The following box has the property that each side-length is an integer and each diagonal across a face is an integer.



Challenge1: Find another example of such a box.

Challenge 2: Unfortunately, the length of the longest diagonal inside this box, $\sqrt{73225}$, is not an integer. No one on this planet currently knows an example of a box with all side lengths and <u>all</u> diagonals integers. For world fame, can you find one?

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HARD STUFF:

WHY DOES THE MULTIPLICATION TABLE TRICK WORK?

The claim is that every Pythagorean triple (a,b,c) with $a^2+b^2=c^2$ and *a*, *b* and *c* sharing no common factor can be written in the form:

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

For some pair of integers *m* and*n*.

e.g. (3,4,5) comes from choosing *m* = 2 and *n* = 1.
(5,12,13) from *m* = 3 and *n* = 2.
(7, 24, 25) from *m* = 4 and *n* = 3.

It is easy to see that if *a*, *b* and *c* are of this form, then we have a Pythagorean triple:

$$\begin{aligned} a^2 + b^2 &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 + n^4 - 2m^2n^2 + 4m^2n^2 \\ &= m^4 + n^4 + 2m^2n^2 \\ &= (m^2 + n^2)^2 \\ &= c^2 \end{aligned}$$

Proving that <u>every</u> triple (a,b,c) – with no common factors – has entries of this form is tricky. Here's a proof:

THEOREM: Suppose $a^2 + b^2 = c^2$ with (a,b,c) a triple of integers with no common factors. Then

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

Proof:

1. The numbers a, b, and c cannot all be even. (Otherwise they have a common factor of two.) So at least one of the numbers is odd.

2. If a and b are both even, then $c^2 = a^2 + b^2$ is even, making *c* even. We cannot have this. So one of *a* or *b* (or both) is odd.

Without loss of generality, let's say that *a* is an odd integer.

3. If b is odd, then $a^2 + b^2 = (2k+1)^2 + (2r+1)^2 = 4\left(k^2 + r^2 + k + r\right) + 2$ for some numbers *k* and *r*. This means that c^2 is a square number that is two more than a multiple of 4. This is impossible!

   <u>Reason</u>: Either *c* is a multiple of four, it is one more than a multiple of four, two more than a multiple of four or three more than a multiple of four. In all these cases, we never get that c^2 is two more than a multiple of four!

$$(2q)^2 = 4q^2$$
$$(2q+1)^2 = 4(q^2+q)+1$$
$$(2q+2)^2 = 4(q+1)^2$$
$$(2q+3)^2 = 4(q^2+3q+2)+1$$

So we must have *a* odd, *b* even making *c* odd.

4. Now: $a^2+b^2=c^2$ which means:

$$b^2 = c^2 - a^2 = (c-a)(c+a)$$

But *b* is even and so is divisible by two. So $\left(\frac{b}{2}\right)^2 = \left(\frac{c-a}{2}\right)\left(\frac{c+a}{2}\right)$ is a valid equation involving integers. (Recall that *a* and *c* are odd.)

Now comes the really tricky part!

5. If $\frac{c-a}{2}$ and $\frac{c+a}{2}$ are both multiples of a common number *k* then their sum, which is *c* and their difference, which is *a*, would also be multiples of *k*. And since $b^2 = c^2 - a^2$, b^2 would be also be a multiple of *k* and AFTER SOME TRICKY THOUGHT ABOUT HOW FACTORS OF NUMBERS WORK, this means that *b* itself would have to be a multiple of *k*. This is impossible since *a*, *b* and *c* have no common factors except $k=1$. So:

$\frac{c-a}{2}$ and $\frac{c+a}{2}$ have no common factors.

6. SOME MORE TRICKY THOUGHT ABOUT HOW FACTORS WORK gives …

Since $\frac{c-a}{2}$ and $\frac{c+a}{2}$ have no factor in common yet their product is a square number (namely $\left(\frac{b}{2}\right)^2$), this can only happen if each of $\frac{c-a}{2}$ and $\frac{c+a}{2}$ are themselves square numbers:

$$\frac{c-a}{2} = n^2$$
$$\frac{c+a}{2} = m^2$$

Adding and subtracting gives …

$$c^2 = m^2 + n^2$$
$$a^2 = m^2 - n^2$$

Solving for b^2 gives:

$$b^2 = c^2 - a^2 = 2mn.$$

We're done! □

COMMENT: The tricky number theory used here is based on the following principles:

1. Every number breaks down into a product of primes.

Thus when consider a common factor *k* of a set of numbers, it suffices to assume we are considering a factor that is a prime number.

2. Primes *p* has the property that if a product MN is a multiple of *p*, then either M is a multiple of *p* or N is (or both).

Thus, if $R^2 = MN$, with M and N sharing no common factors, then any prime factor *p* of R either goes into M or N (but not both). And since p^2 goes into R^2, then this means that p^2 actually goes into either M or N.

Also, all the prime factors of M and of N go into R^2 and hence R.

All in all, this means that the prime factors of M must come squared making M a square number, and the prime factors of N come squares making N a square number if we hope $R^2 = MN$ to be true.

FINAL COMMENT: All of this tricky number theory is explained with greater detail and greater clarity in Volume 2 of ***THINKING MATHEMATICS!*** available for purchase from www.jamestanton.com.