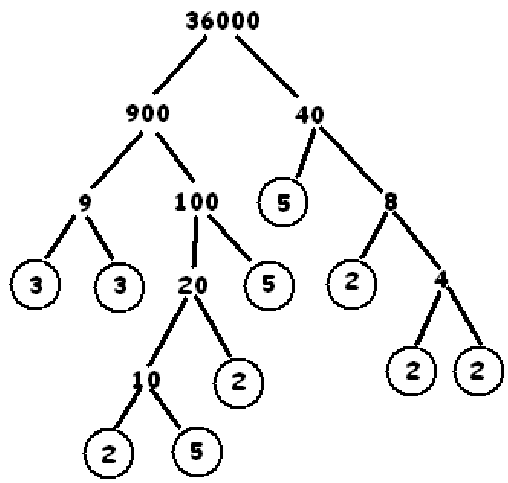
ON FACTOR TREES

In grade school one draws factor trees. For example, here is a tree for the number 36,000:



At each stage one splits the number at hand into a pair of factors, halting at the primes. (This forces the tree to stop. Good thing that 1 is <u>not</u> considered prime!) The tree allows us to then write the starting number as a product of primes. Here, for instance, we see $36000 = 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5\cdot 5$. What is astounding (though most people don't seem to think so) is that despite all possible different choices one can make along the way each tree for any given number produces the same list of primes in the end. (Thus, for example, if we started this factor tree with $36000 = 360\times 100$ we'd still, allegedly, obtain five 2s, two 3s and three 5s in the end.) It is not at all obvious that matters should work out this way! (Read on for indication of what can go wrong.)

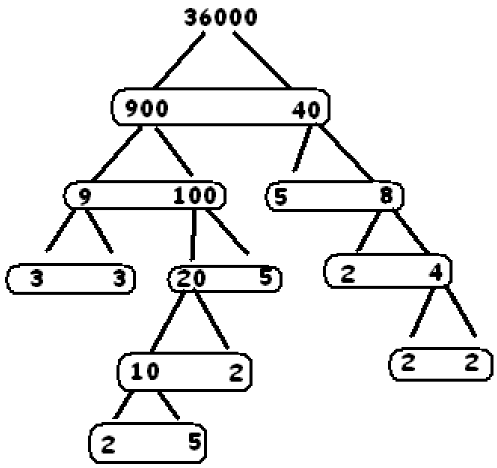
SOME LITTLE-KNOWN FACTS ABOUT FACTOR TREES:

Let's play first and discover these facts along the way:

a) Draw a different factor tree for 36000 (do it – you'll need it for what comes) and verify that the same primes do seem to appear.

Factor trees hold other surprising invariants.

b) The factor tree I drew contains nine pairs of numbers:



I bet the factor tree you drew for a) also contains nine pairs. Why must all factor trees for a *given* number contain the same number of pairs?

b) In the example above, there are four pairs with both numbers even. I bet the same is true for the factor tree you drew. Why must each factor tree give the same number of even/even pairs?

c) In my tree there are two pairs with both numbers divisible by 5. How do I know that the same must be true for your tree?

d) The product of the numbers in a pair is, of course, the number just above the pair. (900×40, for example, is 36000 and 9×100 is 900.) Let's multiply one less than each number in a pair and sum the results. In my tree this gives:

$$899 \times 39 = 35061$$

$8 \times 99 = 792$	$1 \times 3 = 3$
$4 \times 7 = 28$	$9 \times 1 = 9$
$2 \times 2 = 4$	$1 \times 1 = 1$
$19 \times 4 = 76$	$1 \times 4 = 4$

SUM = **35,978**

Do the same for your tree and verify that you obtain the same sum! Why must this be the case?

[I'll reveal all the explanations at the end of this essay.]

ON UNIQUENESS OF FACTOR TREES

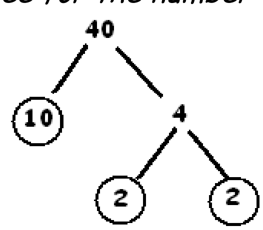
We are led to believe in grade school that all factor trees decompose a given number into the same set of primes, no matter the choices one makes along the way. This is not at all obvious! Suppose Lulu and Gary each agree to devote ten hours to factoring the number 4542847263897240000002300100000000. In what way is it transparent that they each must obtain exactly the same list of primes in the end? Here's a cute example to prove my point:

In the country of Evenastan only even numbers exist! If you ask a citizen of that land to count to ten, she will respond: 2, 4, 6, 8, 10. (And if you ask her to count to 11, she'll only give you a puzzled look. There is no such thing as "11" in Evenastan.)

In this world of evens, some numbers factor and some don't. For example, 24 factors (4×6) but 26 does not. (Remember, "13" does not exist.) Those numbers that factor are called <u>e-composite</u> (short for "Evenastan-composite") and those that don't <u>e-prime</u>.

Exercise: List the first twenty e-primes.

Just as for the U.S., young children in Evenastan are taught to draw factor trees. For example, here is a factor tree for the number 40:



This shows that 40 factors into e-primes: $40 = 2 \times 2 \times 10$

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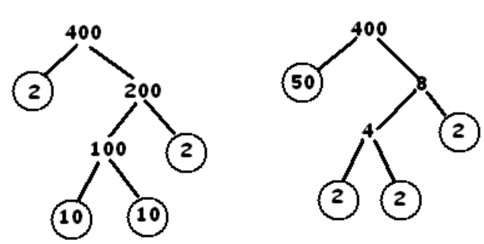
But unlike the children in the U.S., young Evenastan scholars realize that factor trees are <u>not</u> unique. For example, here are two different factor trees for the number 400 showing that this number decomposes into e-primes in at least two different ways.



400 = 2x2x10x10　　　400 = 2x2x2x50

Challenge: *What is the smallest number in Evenastan that factors into e-primes in more than one way?*

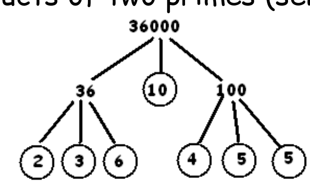
So … If factor trees are not unique in Evanastan, what makes us think they are unique in our world of evens and odds?

The great Greek mathematician Euclid (ca 300 BCE) pondered questions of both geometry and number theory and was the first to realize that there is something significant and deep to prove about our counting numbers. Although the details are beyond the space of this essay (but they are not beyond the space of chapter of chapter 10 of *THINKING MATHEMATICS! Volume 1* available for purchase), let me just say that Euclid did manage to establish that prime factorizations from factor trees are indeed unique for our special arithmetic. What we were implicitly told to assume true is actually true!

It is very easy to train our students - and ourselves! – to take a passive role, not to query "accepted norms" and not to even think to question mathematics. It is okay to have a burning question at the back of one's mind to ponder upon over weeks, months, even years. Mysteries can be joyous and finally resolving them … euphoric! This is what makes math an intensely human experience. Much praise to Euclid for allowing himself to ask and to wonder, and to experience the true joy of mathematics!

Challenge: Do factor trees in Oddvanistan seem to be unique?

RESEARCH CORNER: Suppose factor trees are based on triples rather than pairs, stopping at primes and at products of two primes (semiprimes!). For example:



These trees certainly are not unique in our arithmetic. Any invariants nonetheless?

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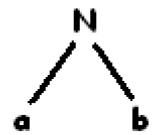
And finally …

EXPLANATIONS:

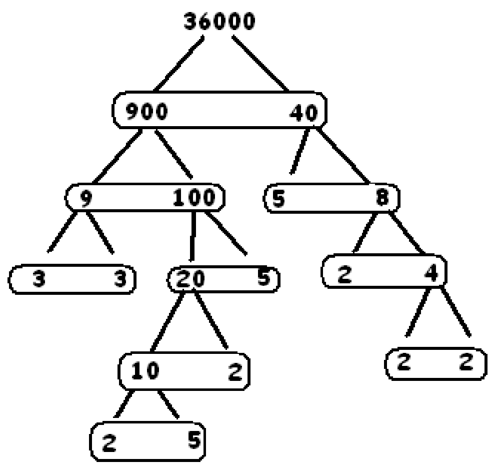
1. Let $A_1, A_2, A_3, \ldots$ be any sequence of numbers of your choosing. For each pair in a factor tree:



(here $N = ab$) write the quantity:

$$A_a + A_b - A_N .$$

If we do this for each pair in a factor tree and sum the result, we see that the answer is independent of the choices made along the way. For example, with



we obtain …

$$\left(A_{900} + A_{40} - A_{36000}\right) + \left(A_9 + A_{100} - A_{900}\right) + \left(A_5 + A_8 - A_{40}\right)$$
$$+\left(A_3 + A_3 - A_9\right) + \left(A_{20} + A_5 - A_{100}\right) + \left(A_2 + A_4 - A_8\right)$$
$$+\left(A_{10} + A_2 - A_{20}\right) + \left(A_2 + A_2 - A_4\right) + \left(A_2 + A_5 - A_{10}\right)$$

There are plenty of cancellations leaving:

$$A_2 + A_2 + A_2 + A_2 + A_2 + A_3 + A_3 + A_5 + A_5 + A_5 - A_{36000}$$

These are the terms corresponding to the primes of 36000 and the number 36000 itself. Notice that all the intermediate terms have vanished and this final result is independent on the details of the factor tree itself. <u>All</u> factor trees give this same final result.

Now it is a matter of choosing interesting sequences to play with.

b) Work with the sequence $1,1,1,1,\ldots$ (that is, $A_a = 1$ always). Then the quantity $A_a + A_b - A_{ab}$ equals 1 and so by writing down these quantities and summing, we are just counting the number of pairs in a tree. The final result is $A_2 + A_2 + A_2 + A_2 + A_2 + A_3 + A_3 + A_5 + A_5 + A_5 - A_{36000} = 9$ for all trees!

c) Work with the sequence $A_a = \begin{cases} 0 & \text{if a is even} \\ 1 & \text{if a is odd} \end{cases}$. Then $A_a + A_b - A_{ab}$ equals 0 if both *a* and *b* are even, and equals 1 otherwise. Thus we are counting pairs that contain at least one odd number. The final result must be $A_2 + A_2 + A_2 + A_2 + A_2 + A_3 + A_3 + A_5 + A_5 + A_5 - A_{36000} = 5$. As the number of pairs is 9, this means that there must be 4 even/even pairs.

d) Work with the sequence $A_a = \begin{cases} 0 & \text{if a is a multiple of 5} \\ 1 & \text{otherwise} \end{cases}$.

COMMENT: There are other ways to answer parts c) and d) noting that the prime factorization of 36000 contains five 2s and three 5s which each must, eventually, be "pulled out" one at a time.

e) Work with the sequence $A_a = 1 - a$. Then $A_a + A_b - A_{ab}$ equals $(a-1)(b-1)$. The sum of these strange products must be invariant!

2. The first twenty e-primes in Evenastan are: 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78.

The number 36 is the smallest number that factors two different ways: $36 = 6 \times 6 = 2 \times 18$

3. Yes! Factor trees are unique in oddvanistan. Since the primes in Oddvanistan are also primes in our arithmetic (they are our odd primes) Euclid's result applies to the primes of Oddvanistan and so they too have unique factor trees!