A BRIEF HISTORY OF LOGARITHMS

For the mathematics and pedagogy of logarithms (and exponents), see:

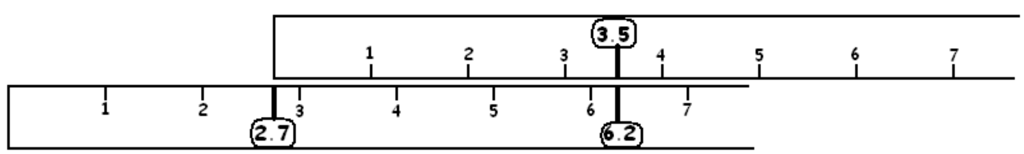
> ***THINKING MATHEMATICS!***
> Volume 1: **Arithmetic = Gateway to All**
> *Chapter 13.*

Let's begin with an activity:

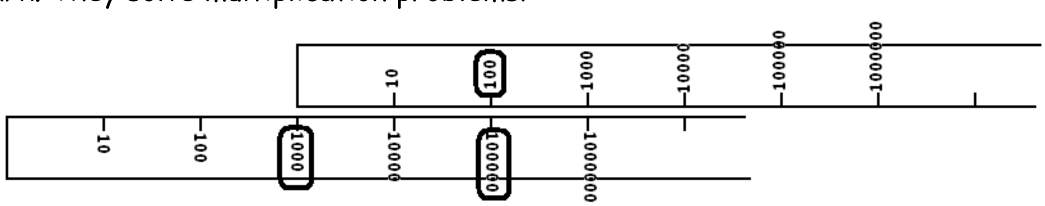
ACTIVITY

Here is how two ordinary rulers solve addition problems:

To compute $2.7+3.5$ *, for example, place the end of ruler 1 at the 2.7 mark of ruler 2. Locate the position 3.5 on ruler 1 and read off the corresponding mark on ruler 2. This is the sum.*



Now consider two rulers with markings given by the powers of ten at each inch mark. They solve multiplication problems!



As a ridiculously simple example, let's compute 1000×100. Place the start of ruler 1 at the 1000 mark of ruler 2. Look for the mark 100 on ruler 2 and read off the corresponding position on ruler 1. This is the product.

a) Cut out two strips of paper and mark the inch positions 10, 100, 1000, …. Use your multiplication rulers to read off a good estimate to the product 87×980. Also estimate the product 120×9903.

b) Why do these rulers work?

THE STORY OF LOGARITHMS:

During the Renaissance in Europe extraordinary advances in the arts and sciences led to new and profound understandings of the natural world. With the invention of the telescope (Galileo first called the device a *perspicillum*) the workings of the heavens were revealed and astronomy took off as an active and fruitful pursuit.

As methods of recording data become more precise astronomers soon found themselves burdened by the simple process of arithmetic when performing calculations. These were, of course, all completed by hand. Adding sets of numbers is not too cumbersome (calculating $3.786 + 5.419$ by hand is not too onerous) but the multiplication of multi-digit numbers is extraordinarily tedious and prone to error. (Try computing 3.786×5.419 by hand. Now imagine computing the product of twenty such numbers by hand!)

In the late 1500s, Scottish mathematician John Napier (1550 – 1617) took it upon himself to help out his beleaguered scientific colleagues. He set out to devise a method that would convert multiplication problems into simpler addition problems. He succeeded, but his approach was creative to say the least.

Napier envisioned two objects moving along a section of a straight line, one for each of two numbers to be multiplied. Their velocities were related, in a complicated way, to the original two numbers and he found that the process of computing the ratio of those velocities in effect converted the process of multiplying the two original numbers into an addition problem! He based the name of this technique on the Greek words *logos* for *ratio* and *arithmos* for number, hence the name *logarithm.* Going further, he decided it would be most helpful to scientists to base his theory on a number relevant to the size of the Earth, namely, $10^7 = 10,000,000$. He chose the number $b = 1 - \frac{1}{10^7}$ as the base of his logarithms and multiplied all the quantities he worked with by 10^7 to help avoid the appearance of decimals. Today his logarithm of a number *N* would be written $10^7 \log_{1-\frac{1}{10^7}} \left(\frac{N}{10^7} \right)$.

After Napier published his work in 1614, English mathematician Henry Briggs (1561-1630) suggested to Napier that, like our number system, logarithms should be based on the number 10. Napier agreed that this would indeed simplify matters and $b = 10$ was then deemed the preferred base for logarithms. Base 10 logarithms are today called common logarithms or Brigg's logarithms. The common logarithm of *N* is simply denoted $\log N$.

Despite the complication of this theory, Napier's methods were tremendously successful and highly praised. He developed tables of values so that practitioners need not worry about matters theoretical and could simply consult booklets he devised. (For example, to compute 2.3×5.7 one consults tables to see that Napier's value for 2.3 is o.43, say, his value for 5.7 is 0.76, adds these by hand: $0.43 + 0.76 = 1.19$, and return to the tables to see that 13.1 has logarithm 1.19. This is the product.)

English mathematician William Oughtred (1575-1660) realized that two sliding rulers, with labels placed in logarithmic scale will physically perform the addition of logarithms and thus allow one to simply read off the result of any desired multiplication. "Slide rules" were invented.

OUR POOR STUDENTS:

Not to detract from Napier's brilliance, but he missed the obvious! It took over one hundred years for mathematicians to realize that Napier's logarithms are simply exponents backwards! The word "logarithm" is a confusing name for a concept that is actually very simple. Few students have trouble reading a statement such as the following:

$$power_2(32) = 5$$

"The power of two that gives the answer 32 is five." But as soon as we write $\log_2(32) = 5$ clarity and transparency is replaced by horror and fear!

Think about it. Do the following look reasonably un-scary (even though I am choosing some bizarre numbers)?

$$power_4(16) = 2 \qquad power_{10}(million) = 6$$

$$power_5\left(\frac{1}{25}\right) = -2 \qquad power_{\frac{1}{3}}(3) = -1$$

$$power_{0.01}(10) = -\frac{1}{2} \qquad power_7\left(\sqrt{7}\right) = \frac{1}{2}$$

When logarithms are presented to students, it is often done in a way that feels remote and abstract, chiefly because of the notation. (If I were Czar of the mathematics teaching universe, one thing of the three things I would first decree is that all appearances of "*log*" in mathematics textbooks be replaced with "*power*" – with multiple side comments reiterating no disrespect to Napier.)

NAPIER'S DREAM:
Do we indeed have $power(M \times N) = power(M) + power(N)$ as Napier worked so hard to obtain? Well, yes!

For simplicity, let's work with base 2 and check if there is reason to believe $power_2(8 \times 16)$ should equal $power_2(8) + power_2(16)$. Here goes:

8 is a product of three twos ($power_2(8) = 3$),
16 is a product of four twos ($power_2(16) = 4$),
8×16 is the product of $3+4$ twos.

That is, it takes $3+4$ twos to "make" 8×16.

That is, $power_2(8 \times 16)$ really does equal $3+4$, which is $power_2(8) + power_2(16)$.

In general – and informally…

It takes $power_2(M)$ twos to make M,
It takes $power_2(N)$ twos to make N ,
and so it takes a total of $power_2(M) + power_2(N)$ twos to make $M \times N$.

COMMENT: To make this argument proper and formal, note that

$$power_2(M \times N) = power_2(M) + power_2(N)$$

is saying that $power_2(M) + power_2(N)$ is the power of 2 that gives the answer $M \times N$. To check we need to use it as a power to two and see if we do indeed obtain the answer $M \times N$.

Note first that $2^{power_2(M)} = M$ (since $power_2(M)$ <u>is</u> the power of two that gives M) and $2^{power_2(N)} = N$.

Now we are ready. Here goes:

$$2^{power_2(M)+power_2(N)} = 2^{power_2(M)} \cdot 2^{power_2(N)} = M \cdot N$$

Yes! $power_2(M) + power_2(N)$ is indeed the correct power of two to give $M \times N$.

CHALLENGES:

EXERCISE: Explain each of the following "log" rules:

$$power_3(M \div N) = power_3(M) - power_3(N)$$

$$power_{10}(10^x) = x$$

$$b^{power_b(x)} = x$$

$$power_b(1) = 0$$

$$power_b(b) = 1$$

$$power_b(b^x) = x$$

COMMENT: In the notation of logarithms, these rules translate as follows:

$$\log_3(M \div N) = \log_3 M - \log_3 N$$
$$\log(10^x) = x$$
$$b^{\log_b x} = x$$
$$\log_b(1) = 0$$
$$\log_b(b) = 1$$
$$\log_b(b^x) = x$$

EXERCISE: I tell you that, for some number *b*:

$$\log_b 2 = 0.693$$
$$\log_b 3 = 1.098$$
$$\log_b 5 = 1.609$$

Without a calculator, find $\log_b 4$, $\log_b 6$, $\log_b 8$, $\log_b 9$, $\log_b 10$ and $\log_b 600$. Estimate $\log_b 7$ and $\log_b 70$. Also estimate $\log_b(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10)$.