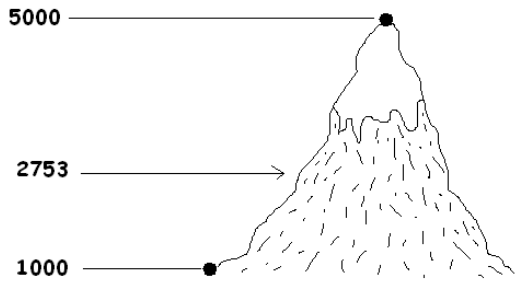
JOYS OF THE INTERMEDIATE VALUE THEOREM

Further discussion on this topic appears in:

THINKING MATHEMATICS! Volume 6: **Calculus** *Chapter 3*

THE INTERMEDIATE-VALUE THEOREM: If I climb a mountain starting at an elevation of 1000 feet, ending at an elevation of 5000 feet, must I have passed through an elevation of 2753 feet?



YES! Of course! - assuming we are not talking of fanciful situations in which folk can magically transport from one location to another without passing through all points in between.(**Question:** Might I have passed through that elevation more than once?)

This very simple idea is surprisingly powerful. In mathematics it is called the Intermediate-Value Theorem:

> *Any continuous process that starts at one value and ends at another must pass through all values in between.*

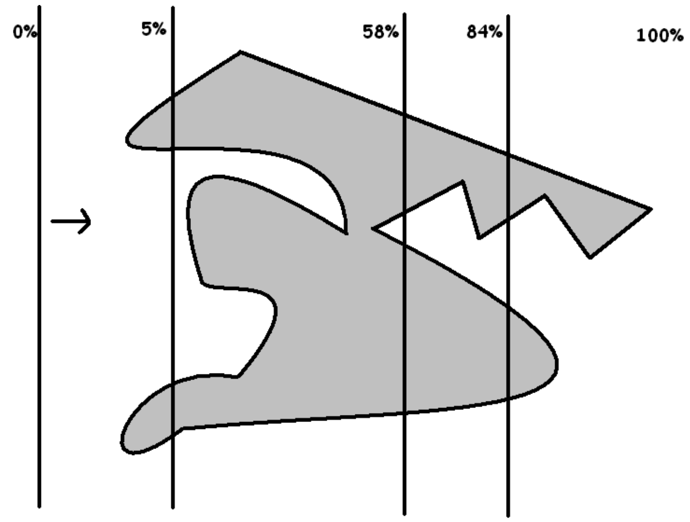
This essay explores some astounding consequences of this very simple idea.

COMMENT: A typical first course in calculus course gives a formal statement of this theorem, without proof, and uses it to "prove" that solutions to equations exist. For example, to show that a there is a value x for which $x^3+3=3^x$, let $f(x)=x^3+3-3^x$. Then $f(0)=0+3-1=2$ and $f(4)=64+3-81=-14$. One then argues that this continuous function must pass through the output of zero as we "move" from an output of 2 to an output of -14. Thus, a solution to the equation exists.

This is a dry – and somewhat dull – application of this beautiful result. In this essay we'll explore the not so dry and not so dull consequences!

1. **THE ONE PANCAKE THEOREM**: *Given a blob fixed in the plane (an irregular pancake), there exists a vertical line that divides its area exactly in two.*

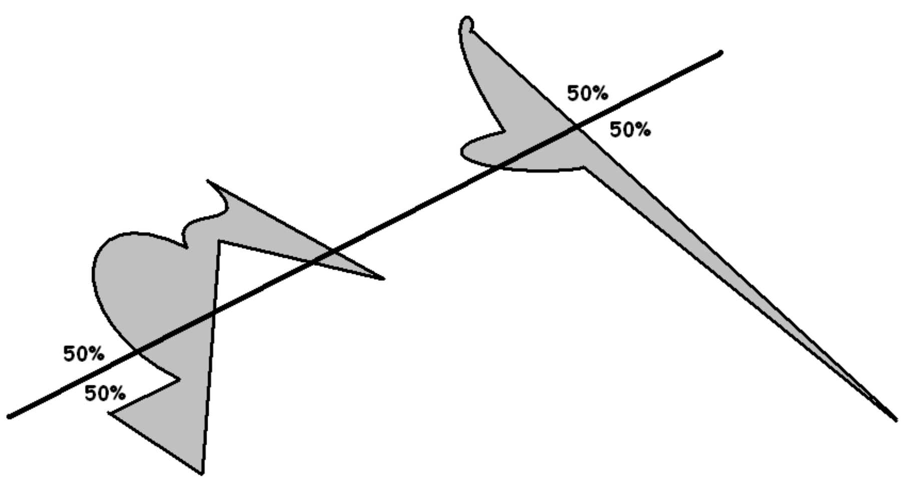
Hold a straight knife to one side of the pancake and slide it across the cake.



Measure the area of the pancake to the left of the knife as it sweeps across. We start with 0% to the left and end with 100% to the left. It seems reasonable to believe that this is a continuous transformation. Consequently, we must have passed through a position with 50% of the area to the left (and 50% of area to the right). This position cuts the pancake exactly in half.

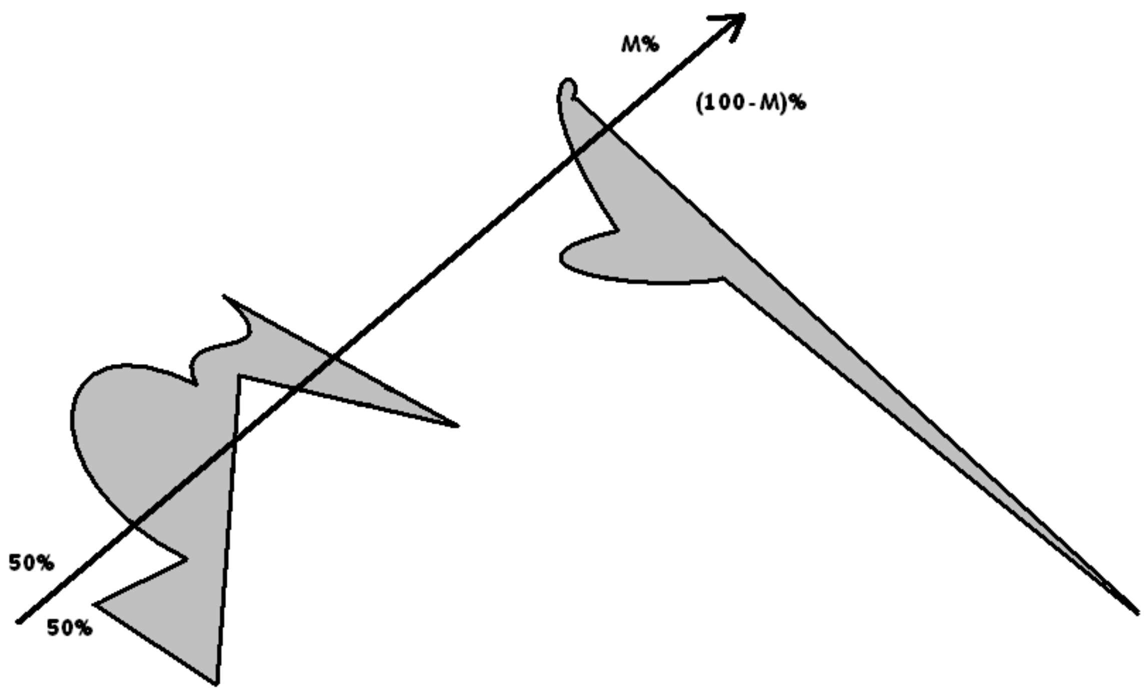
<u>Comment</u>: There is nothing special about vertical lines here. The same result holds for a line given at any prescribed angle.

2. **THE TWO PANCAKE THEOREM:**
Given any two shapes in the plane, fixed in position, it is possible to find a single straight line that simultaneously divides the area of each shape exactly in half.



We know by the one-pancake theorem for each angle θ there is a line at that angle that slices the first pancake perfectly in two. That line, however, might or might not slice the second pancake in half. (It might even miss the second pancake entirely!)

For each angle θ, consider the position that slices the first pancake in half for that angle and let $A(\theta)$ be the area of the second pancake to the left of the knife. Suppose $A(\theta) = M\%$. Do you see that $A(\theta + 180^\circ) = (100 - M)\%$? (It is the same knife cut, but in the reverse direction.)



Notice that if M is less than 50% then $100 - M$ is greater than 50%, and vice versa.

The continuous process of varying the angle θ (always cutting the first pancake in half, recording the percentage of the second pancake to the left of the knife) passes between the values M% and (100-M)% and so must pass through the intermediate value of 50%. This knife position does the trick!

3. **THE PIZZA THEOREM:** *For any given irregular pizza there is a single straight cut that divides the area and the perimeter of the pizza each exactly in half.*

This result is essentially the two-pancake theorem in disguise. The second "pancake" is just the thin curve that is the crust of the pizza.

4.**THE EQUATOR THEOREM:** *At any instant there exist two points directly opposite one another on the Earth's equator with exactly the same air temperature.*

Identify each point on the equator by its longitude θ, and for each angle θ, let $f(\theta)$ equal "air temperature at position θ minus air temperature at position $\theta+180^\circ$." (That is, $f(\theta)$ is the difference in air temperatures at opposite points on the equator.)

Notice that as θ varies between 0° and 180°, $f(\theta)$ changes sign, and so must pass through the value zero.

5.**A FAMOUS PUZZLE:** *In my mountain climb of the previous page, I started my journey from elevation 1000 feet at 7 a.m. in the morning and reached the summit at 5000 feet 7 p.m. that evening. The next day I took a different path back down the mountain to return to my starting point by 7 p.m. Prove there is an exact time somewhere between 7 a.m. and 7 p.m. for which I was at exactly the same elevation on each day at that moment!*

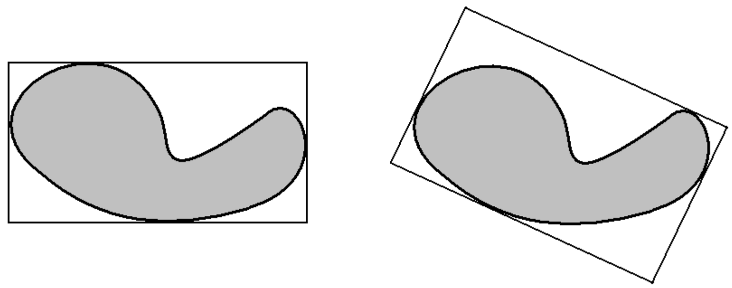
<u>Answer 1</u>: Let $f(t)$ be "my elevation at time t on day one minus my elevation at time t on day two." We have:

$$f(7a.m.) = 1000 - 5000 = -4000$$
$$f(7p.m.) = 5000 - 1000 = 4000$$

There must be some intermediate time with $f(t) = 0$.

<u>Answer 2</u>: As I start down the mountain imagine that my friend starts at the bottom of the mountain retracing exactly the movements of my journey up. What can I say about the moment we pass each other at some common height?

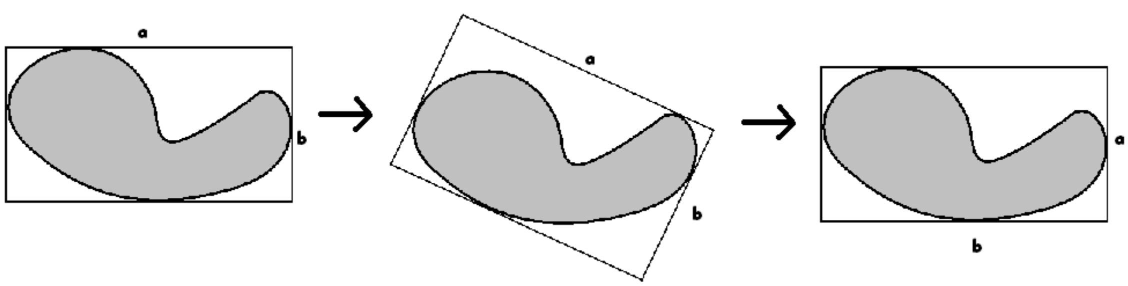
6. **CAPTURE THE BLOB:** *Here is a blob captured inside two rectangles, one with vertical and horizontal sides, and one with sides at some angle. (Notice that in each case the blob touches each of the four sides of the rectangle. Each rectangle is thus the smallest possible rectangle aligned in the given way that "captures" the blob.)*



Is it possible to capture this blob in a square? If so, can every blob (assuming it is not infinitely large) be captured in a square no matter its shape?

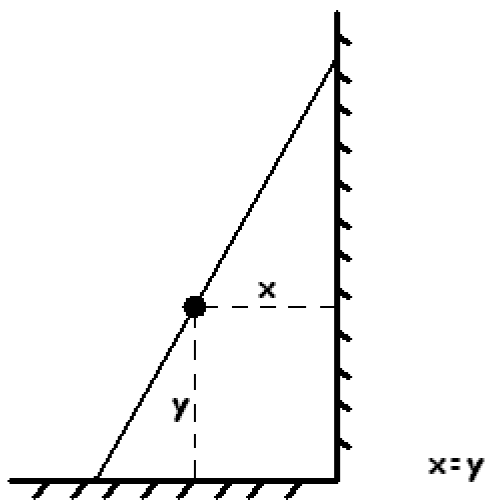
For each angle θ, there is certainly a rectangle that "captures" the blob (any blob) with one side at slope angle θ, (Bring in a line at that angle far away from the blob and slide it until it just touches the blob. Do that for the opposite side of the rectangle and the two remaining perpendicular sides.)

For each angle θ, let $f(\theta)$ be the length of the side at angle θ minus the length of an adjacent side. (That is, in the left image below, $f(\theta) = a - b$.) As the diagram below shows, $f(\theta + 90^\circ) = b - a$. Thus $f(\theta)$ and $f(\theta + 90^\circ)$ have opposite signs. There must be some intermediate angle for which this difference is zero. At this angle, the rectangle is a perfect square.

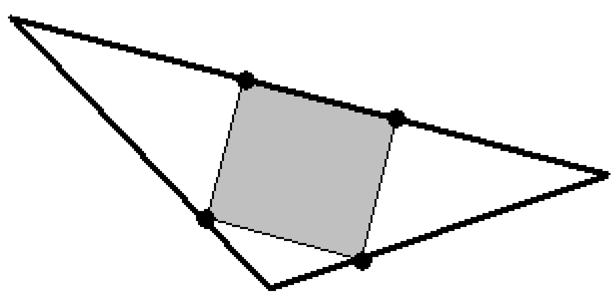


CHALLENGES FOR YOU!

7. *A ladder leans against a wall. Prove that here exists some point on the ladder whose vertical distance from the floor equals its horizontal distance from the wall.*

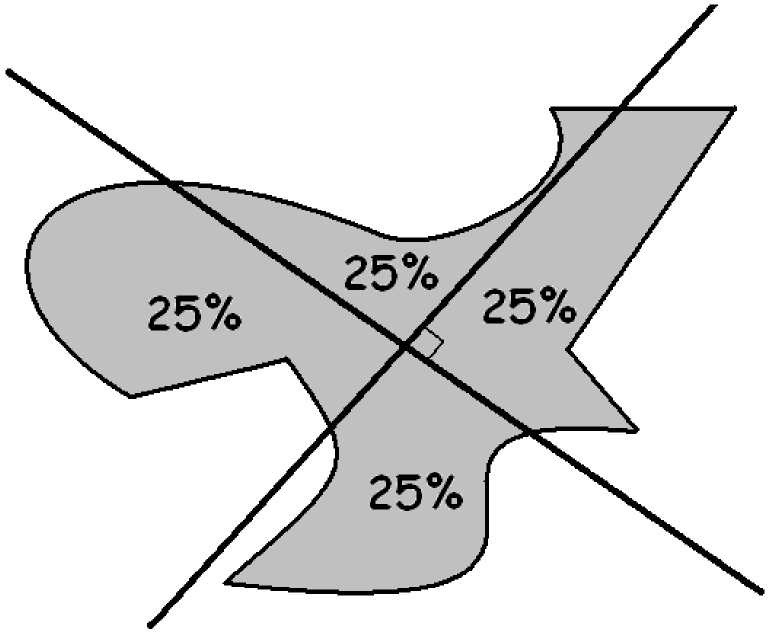


8. *Prove, for <u>any</u> triangle, it is possible to draw inside it a perfect square with each of its four corners lying on the side of the triangle. (Two corners lie on the same side of the triangle.)*



9. *A piece of string is stretched straight from the left wall of a room to just touch the right wall of the room. The string is then crumpled and thrown into the middle of the room. Establish that at least one point on the string is the same distance from the left wall as it was when the string was stretched straight.*

10. **(TOUGH CHALLENGE)** *Establish that for any given shape in the plane there exists a pair of perpendicular lines that divide the area of the shape into quarters*



(This problem is not as straightforward as it first appears!)