WHAT MADE ME A MATHEMATICIAN

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(and why I approach mathematics teaching the way I do)

The following essay appears as the introduction to my **THINKING MATHEMATICS!** series and is printed in *Volume I: Arithmetic = Gateway to ALL.* It explains all!

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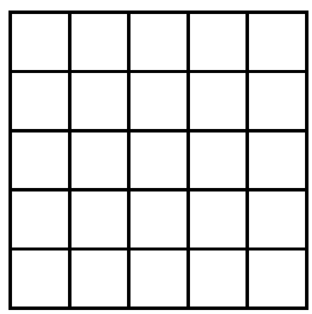
THE STORY BEHIND THIS SERIES

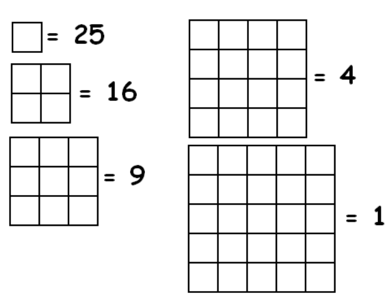
My career as a mathematician and as a teacher of mathematics began at age ten. I didn't realize this at the time, of course, but in retrospect it is clear to me that my journey into the rich world of mathematical play – and I use the word *play* with serious intent – was opened to me thanks to a pressed-tin ceiling in an old Victorian-style house.

I grew up in Adelaide, Australia, in a house built in the early 1900s. The ceiling of each room had its own geometric design and each night in my bedroom I fell asleep staring at what was essentially a 5x5 grid of squares above me. (The lines of the grid were vines and each corner held a floral design.)

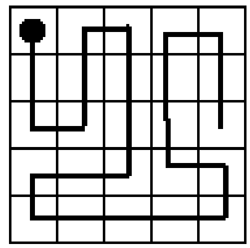


I counted squares and rectangles in the design. I traced paths through its cells and along its edges. I tried to fit non-square shapes onto the vertices of the design. In short, I played a myriad of self-invented games and puzzles on that grid of squares as I fell asleep. And a number of puzzlers have stayed in my mind all these years as particularly rich and curious:

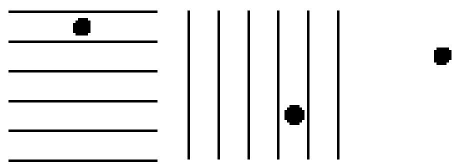
1. Count the number of squares one can find of each size in the 5x5 grid. Is it obvious that each count should be a square number?



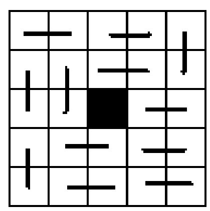
2. Starting at the top left corner and taking just horizontal and vertical steps it is possible to walk a path that visits each and every cell of the grid exactly once:



Is it possible start such a path from <u>any</u> desired cell of the grid? There is something troubling with regard to the third picture below.

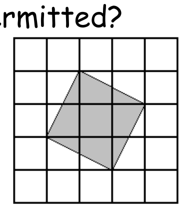


3. It is possible to pair neighbouring cells of the grid and leave the middle cell as the "odd man out."



Can a corner square be an odd man out? Can each square in the grid be singled out as special in this way?

4. How many squares in total can be found in the grid if titled squares such as the one shown are also permitted?



5. Is it possible to draw an isosceles triangle in the grid with corners at grid points? How about an equilateral triangle? Can one draw a square on a triangular array of dots?

I wonder at times if I may have been an unusual child – but I don't really think so.

The nature of *play* – that is, intellectual exploration, intellectual curiosity, the pursuit of wanting to know – is innate to our true human selves. Children of all ages love finding patterns and love pushing patterns to the realm of the extreme. (Look! $1=1$, $1+3=4$, $1+3+5=9$, $1+3+5+7=16$, $1+3+5+7+9=25$, square numbers! A sum of odd numbers seems to be square. Does this mean that the sum of the first million odd numbers is one million squared?) The search for patterns is at the heart of science and scientific research, and of mathematics too. But mathematicians insist on taking matters a step further by exploring and searching for logical rationale and a depth of understanding. Is there any reason to believe that a particular pattern is true? (Is the sum of the first n odd numbers indeed always n^2? How can we know this to be the case?) Mathematicians delight in both proving patterns as valid and also in finding examples where they break down.

We begin our formal mathematical training in grade school and find joy in the realm of arithmetic. We discover the counting numbers and basic operations we can perform on them. Some children revel in computing squares and cubes of numbers, for example, or noticing that multiplying two numbers that end in a 6 produces another that ends in 6, or that no square number ends in an 8. As young students we are often asked to identify patterns and, if we have an interesting and enlightened curriculum, we may be asked to attempt to explain why a pattern is true.

At some point, however, matters often seem to change in the standard mathematics training. Time for exploration and play of ideas is replaced with skill sets and competencies to be met. I remember asking my fifth-grade teacher why negative times negative is positive and receiving no answer other than to accept it as true and being told to move on to the problem set at hand. As a child I learnt that my role in this next level of mathematics education was a passive one and I was not meant to ask questions.

But I did again, once, and I am glad I did. I think this was my second defining life experience as a budding mathematician. In a session on the Pythagorean theorem the teacher asked the students in class – all 35 of us – to each draw three right triangles on a sheet of paper, measure the side-lengths, calling them a, b and c (of course! What other symbols are there for this?) and to perform the appropriate computations to see if "a squared plus b squared equals c squared." I remember all my young colleagues doing this and agreeing that this proved to be the case every time and therefore must be true in general(?). I wasn't convinced and I raised my hand to ask: "How do we know that this isn't just coincidence? Maybe by luck it worked 102 times in a row." (There was a second issue in my mind that I didn't express, namely that I didn't actually believe anyone saw it to be true even once: one cannot measure lengths exactly!) The teacher's response was affecting. He said: "Go back and draw another three right triangles." End of conversation! My suspicion was confirmed: I was on my own with regard to understanding why the Pythagorean theorem might be true. I spent the next couple of years of my life (on and off) trying to figure out why it should be so.

To be honest, I found high-school mathematics quite uninspiring and limiting. My desire to ask questions, to explore ideas, to confront the whys, and to play intellectually, although not squelched, was put on hold. It wasn't until I took a university course in abstract algebra and number theory – the course that, in essence, simply asks the "why" in the arithmetic of high-school mathematics – did I find my true intellectual home and my place to play. In some sense my intellectual self was set free. It was joyous and it was liberating.

But even here I felt like an anomaly: the majority of my classmates were complaining about the course as being too abstract, too disconnected from the "real world," devoid of meaning and, just plain too hard. I didn't understand. Couldn't my colleagues see that, finally, this one single course was explaining why most everything we had learnt with regard to arithmetic and algebra in school was true? Couldn't they now see why factor trees always yield the same answers? Why negative times negative just had to be positive? Why complex numbers make sense and help with real problems? Why Pascal's triangle is connected to expanding brackets? This course wasn't about finding patterns and being satisfied at that, but about explaining why patterns and observations had to be true. It was mathematics!

Upon reflection – and this may be too harsh a judgment – I wonder if some of my colleagues felt too much at a loss as to what to believe and not to believe. "Why do we have to prove something that we already know to be true?" is a refrain I recall hearing more than once. We had been "trained" to believe that many facts are "obvious": If a number is multiple of 2 and of 3, then obviously it has to be a multiple of 6! (So the same is true for the numbers 6, 8 and their product 48?) Dividing fractions is just multiplying by the reciprocal, of course (?). Long division of numbers just works! Perhaps my colleagues had been urged to accept supplied facts without question- just as I had felt obliged to do - and have since found intellectual safety and ease in being told what to believe as true in mathematics. "Why do we have to prove something that we already know to be true?" Answer: Because we, personally, each don't know it to be true in the first place!

The goal of this series is to simply re-examine the standard k-12 mathematics curriculum, starting with matters of arithmetic and algebra, moving on from there, to revel in the delight of intellectual play and not knowing. Is zero even or odd? Is negative zero (if that makes sense) the same as zero? Why is $3^0 = 1$ and $0! = 1$? Why does the divisibility rule for 3 work? What's a divisibility rule for 7? Why is negative times negative positive? What does Pascal's triangle really tell us? Should we trust patterns? What is infinity? What is this thing called "synthetic division" and what is it really doing? Is there such a thing as base one-and-a-half? How many prime numbers are there? Why are primes interesting? Let's find out!

This book is full of explorations (both "hands on" and "intellectual) all designed to entice one into both extending questions and creating questions. Success in both business and scientific research comes from asking questions, exploring and playing with ideas, and being flexible in one's thinking and innovative in one's perspective. Sure, teachers can, and should, teach students a number of skill sets - and this is a valuable and appropriate enterprise. But that should not be the end of the story. As teachers we shouldn't deny students the opportunity to explore the creative aspect of being a mathematician. But this means understanding that creative process ourselves. I hope these books will help in that regard. We want to foster innovation, insight, and flexibility of mind. We want to promote true personal understanding and a desire to seek depth in one's own knowledge. And surely we want the learning of mathematics to be rich and joyous. The typical high-school English curriculum teaches students both the grammar and the poetry. A mathematics curriculum should do so too.

Admittedly, these books assumes the reader – a practicing teacher, a teacher to be, a student, a parent, or a fellow human being interested in mathematics – has already experienced a good portion of the typical k-12 for him- or herself. This book plays off of this assumption and this choice of approach is deliberate for several reasons.

1. A depth of personal understanding comes from only visiting a story multiple times. When one begins a new topic, the bulk of the mental effort is taken up with understanding the jargon, the style of ideas, and the feel for the subject. It can be exciting – and frustrating – as one accustoms one's mind to new ideas and new approaches. However, just because one has read and worked through a chapter on logarithms, say, and can "do" the mechanics of the subject, it usually does not mean that one has internalized the subject and come to personally own and understand the topic. The depths of a subject can only be seen once one has accustomed oneself with the topographical features of the surface. As one of the goals of this text is to bring depth to each topic studied, we need to be ready to dig for it.

2. The book is written to help develop and strengthen personal understanding of mathematics so as to help those involved make better and informed choices for their teaching methods and choices of approaches to adopt with students. Although the details of any particular curriculum vary from school to school and the textbooks used (and consequently it is not possible to

address all curricular directly) we do follow the standard broad curricular themes in this series. These books should thus be deemed as an overview of the story of arithmetic, algebra, functions, calculus, probability, statistics, and so on as presented in the school world. We do not offer advice or recommendations on how to teach a particular topic (though plenty of specific teaching ideas may come to mind in reading this text). The goal is simply to help a teacher – or parent - develop and strengthen his or her own personal understanding so as to make informed, personal pedagogical choices, and to learn how to handle questions that arise in the classroom. Students reading the text will find depth to the material they have studied in school, as well as a good hint as to the working mind of a practicing mathematician. Overall, this book's goal is to take familiar concepts to new and exciting heights.

3. It is easy relatively easy to "do" mathematics. But innovation and initiative requires moving beyond the mere "doing" towards the creative and the inventive. A leading-edge company seeks to push boundaries, to create and ask new questions, to pursue new ideas, and to garner new perspectives on the familiar. Scientific research works towards the same. How do we teach the art of asking questions? Focusing on details of a particular curriculum, although important, might not help with the issue of general perspective. In this series we choose to emphasize broad themes as a means to illustrate "perspective" and offer context and a means to explore exciting, intellectual realms – even within the familiar.

4. Another goal of this series is to simply portray the mindset of a practicing mathematician, to share the beauty of mathematics, and to revel in the sheer joy of exploring this wondrous subject. Having already experienced much of the standard k-12 mathematics curriculum, the reader has been handed a knap-sack of tools with which to play. We are equipped to play!

So … On that note, let's get cracking! However you choose to use these books – by reading and mulling carefully from beginning to end, or by flipping through and picking up on tidbits that catch your eye – I hope you find the experience rich, intriguing, enlightening, helpful, and, most of all, as studying and doing mathematics should be, joyous! The grammar and the poetry, together.

-JST 2008

p.s. Some hints?

1. To count the number of 2×2 squares, for example, that lie within a 5×5 grid, look at the possible locations of the bottom-left corner of each small square. Do you see a 4×4 array of possible locations?

2. Color the cells of the grid black and white in a checkerboard design with the top left corner black. There are then 13 cells colored black and 12 white. A path that starts on a white cell will "use up" all the white cells before all the black cells are reached.

3. Use the same checkerboard coloring again. Each pair of cells "uses" one cell of each color.

4. Each tilted square sits "snugly" inside a non-tilted square. Perhaps reverse this notion and ask instead: How many tilted squares lie snugly within each type of non-tilted square?

5. Isosceles triangles can be found, but no an equilateral triangle is present. This is a tricky to see. First think about the area of any triangle you can draw with vertices on corners of the grid. Can you see that the area of any such triangle is either a whole number of units of area or a whole number plus half a square unit?

THINKING MATHEMATICS!

A Refreshingly Clear Reference
Series for Teachers and Students and all those seeking
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