Pick's Theorem – and Beyond!

The following middle- and high-schoolers from the St. Mark's Institute of Mathematics Spring 2009 research class (www.stmarksschool.org/math) developed this material. A shortened version of this article is to appear in FOCUS, published by the Mathematical Association of America:

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The article:
In 1899 George Pick discovered a relationship between the area A of a simple lattice polygon (that is, one whose vertices have integer coordinates), the number b of lattice points on the boundary of the polygon and the count i of lattice points inside the polygon. He showed:

$$A = i + \frac{b}{2} - 1$$

Folk typically prove this formula by subdividing the polygon into lattice triangles after having verified two things: the result is true for all lattice triangles and the result remains true if a triangle is "attached" to polygon for which the formula already holds. The greatest difficulty lies in establishing the result for triangles.

In the Spring of 2009 young students (ages 9 -17) attending the St. Mark's Institute of Mathematics research group took another approach to Pick's theorem. They began by questioning the coefficients that appear in the formula: Why are interior points each "worth" 1? Why are boundary points each worth ½? Examination of a lattice rectangle leads to an insight.

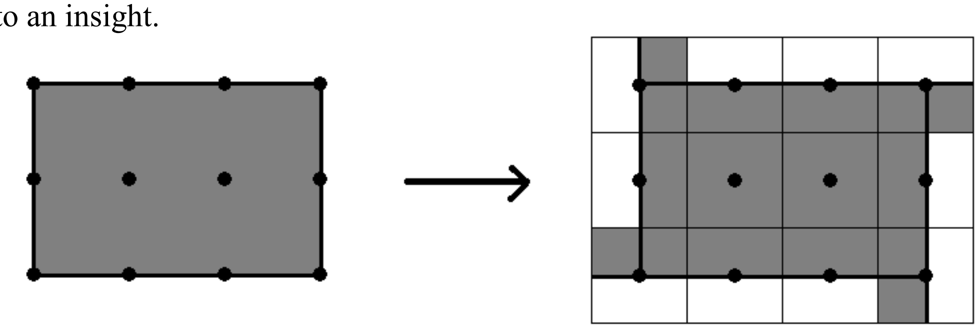


Figure 1

If we surround each lattice point with a unit square with sides at half-integer coordinates (let's call these squares *cells*) then we see that each interior point "contributes" one full square unit of area and each boundary point different from a vertex half a unit of area. If we extend the sides of the rectangle to make its exterior angles explicit we can introduce additional area so that each vertex also contributes half a unit of area. As the exterior angles of any polygon sum to one full turn, this excess in area amounts to one full square unit. The "-1"in Pick's formula compensates this.

This analysis applies directly to any simple polygon with sides parallel to the axes of the lattice (provided exterior angles turned in opposite directions are deemed opposite in sign).

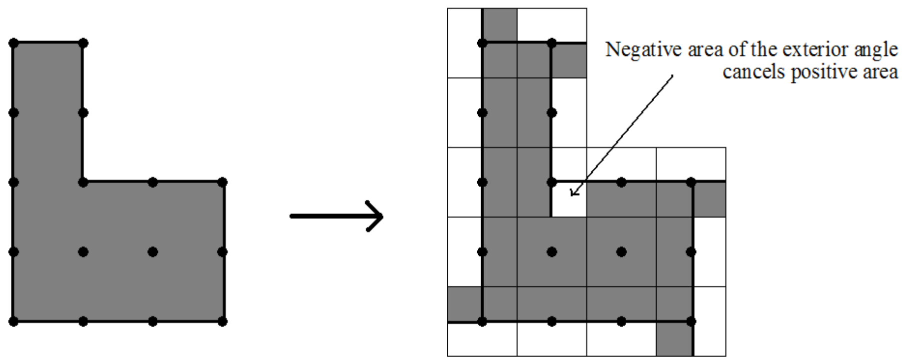


Figure 2

And essentially the same rationale applies to any simple lattice polygon! The key is to note that diagonal line segments connecting two lattice points are rotationally symmetric about their midpoints. In particular, any cell that is intercepted by such a diagonal and divided into two parts is matched by a rotationally symmetric cell divided into the same two parts. (Also note that the matching portions are on alternate sides of the diagonal.)

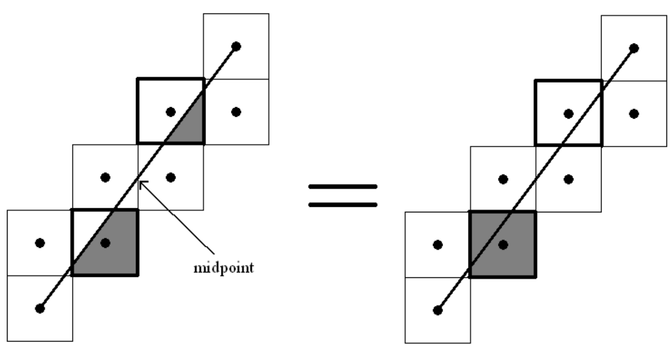


Figure 3

Each subdivided cell with center an interior point of the polygon can thus be "completed" by switching rotationally symmetric portions. (And the analogous result holds for each cell with center in the exterior of the polygon.) We are close to a Proof Without Words of Pick's result:

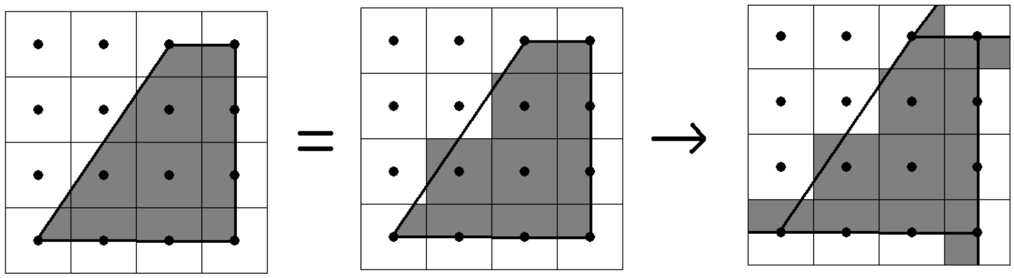


Figure 4: Each interior point contributes one unit of area and each boundary point half a unit of area with an overall “error” of -1.

A complication arises when more than one diagonal passes through the same cell. This can be handled by switching portions about one diagonal at a time. First choose a portion within a cell on one side of a diagonal that is free from other intercepting diagonals. If its rotationally symmetric counterpart is also free from intercepting diagonals, perform the switch. If not, work with a smaller part of the matching cell and attempt a switch there. As there are only finitely many regions to consider, there is sure to be a first switch to perform and all maneuvers thereafter will fall into place.

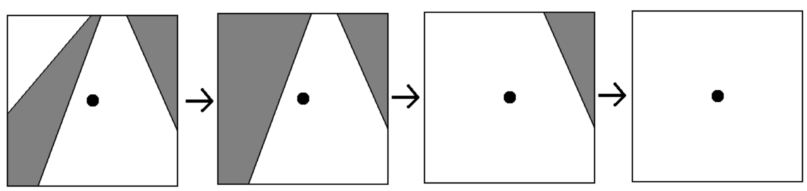
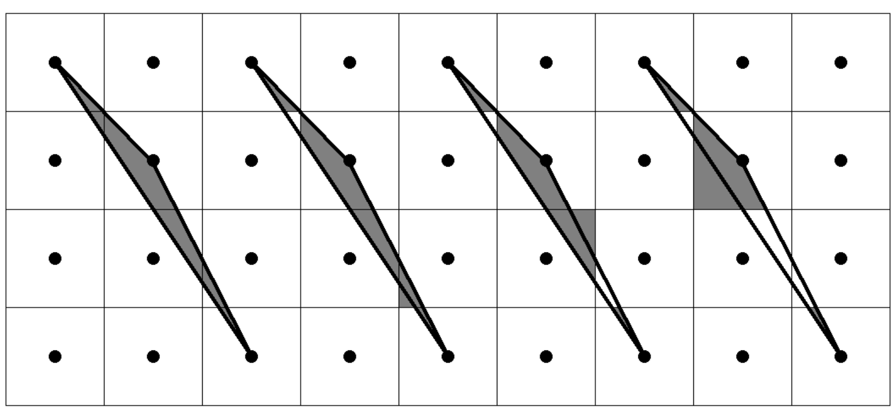


Figure 5. A series of switches.

This technique shows, for example, that the area of the following triangle is one half:



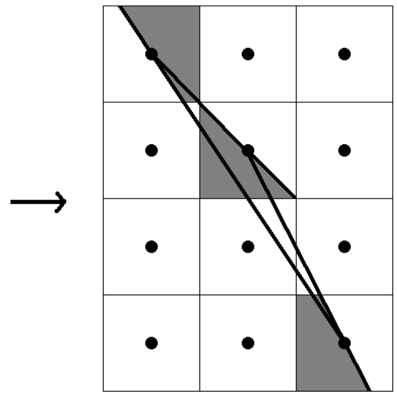


Figure 6. The area of any lattice triangle lacking interior points is $1/2$.

We have established Pick's theorem.

Generalising the result

Of course youngsters want to take a result and twist it in new and intriguing ways. (Just as is the wont of mathematicians!) Does a version of Pick's theorem hold for polygons with holes? For polygons with tendrils? For disconnected shapes?

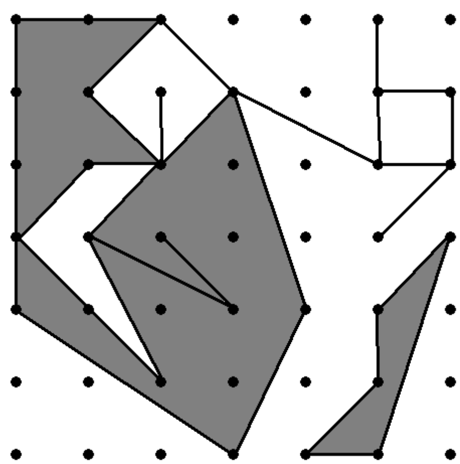


Figure 7: A wild "polygon." Shaded regions indicate area to be evaluated.

As ordinary Pick's theorem assigns weights to points of the polygon (a weight of 1 for interior points and a weight of ½ for boundary points), students developed a general system for assigning weights to points of any wild shape. First take note of the shading. If an edge has the same coloring on each of its sides, thicken that edge slightly and produce either an infinitesimal amount of space or an infinitesimal amount of area wedged between two infinitesimally close edges.

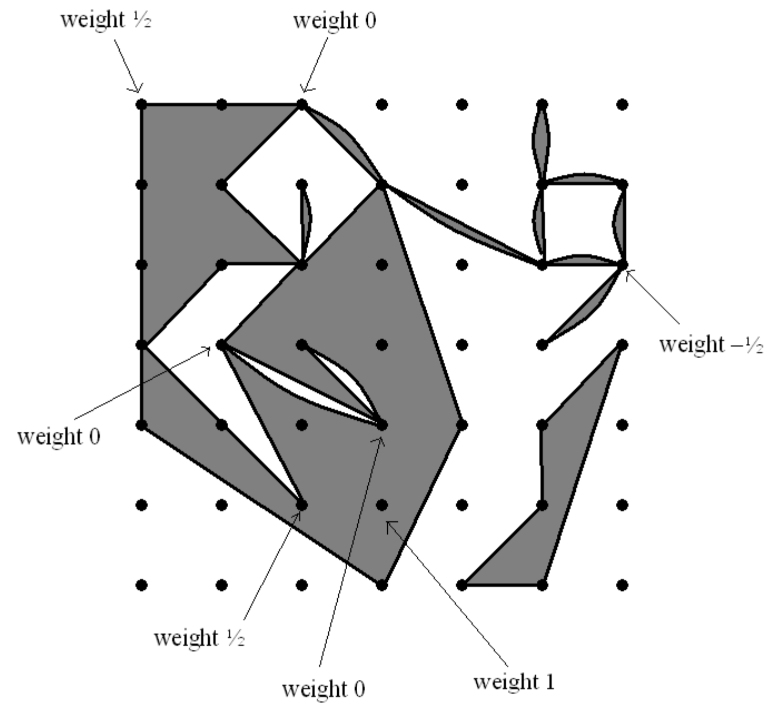


Figure 8

For each lattice point considered part of the shape assign a weight of $1-\frac{e}{4}$ where e is the number of edges emanating from that point. Every other lattice point in the plane is given weight zero. Let w be the total sum of all the weights. For the shape above, $w = 11½$.

Let h be the number of "holes" that appear in the thickened shape (including the infinitesimally small holes), c the number of components of the shape and A the area of the shaded regions (disregarding infinitesimal regions, that is, A is the area of the original "unthickened" shape). In our example, $h = 5$, $c = 2$ and $A = 14½$. Students discovered:

Generalized Pick's Theorem: $A = w + h - c$.

For a simple polygon, $w = i + b/2$, $h = 0$ and $c = 1$ and we have Pick's statement.

Outline of Proof: Consider the original polygon (with no thickened edges) with all shading removed. Using an induction argument on the number of edges connecting a fixed set of vertices, it is straightforward to show that $w + h - c = 0$. (One must still consider thickened edges to correctly determine weights.) This is just the Euler-Descartes formula in disguise. Draw in additional edges to connect the components of the graph (so that c becomes 1) and to make each region a simple polygon. We still have $w + h - c = 0$. Now return the shading of the original shape one region at a time. A little thought shows that the weight of each boundary point of that region (suppose there are b of them) increases by ½ and the weight of each interior point of that region (say there are i of them) increases by 1. The number of holes of the modified shape decreases by 1. Thus we have:

$$A \to A + \text{area of new shaded region}$$
$$w \to w + i + b/2$$
$$h \to h - 1$$
$$c \to c$$

By Pick's original theorem, the formula $A = w + h - c$ thus remains valid at each step. At the end of this process remove the additional edges one at a time. One can see that this too does not disrupt the validity of the formula.

Conclusion: What mighty clever youngsters! The extra-curricular research classes I offer through the St. Mark's Institute of Mathematics are conducted in the style of a conversational Math Circle, that is, through the give-and-take of shared discussion and exploration. My role in these sessions is never to instruct, but rather to guide, nudge and offer my opinions and hunches only when asked (perhaps). I do not know what the outcomes of these sessions shall be. To learn more about Math Circles, visit the MAA website www.maa.org and click on SIGMAA MCST (Math Circles for Students and Teachers).