FOIL or FILO or IFLO or OLIF or …

Do mnemonics help or hinder? I don't know that the answer to that question (and I am sure that the context in which the mnemonic is employed is a relevant factor to answering that question) but I do know that mnemonics can certainly mask student understanding, well, lack of understanding.

In beginning algebra students are often taught to FOIL expressions of the sort $(x+2)(x+6)$. And many students who will happily (?) foil away all sorts of expressions of this ilk.

(**Question:** Is this true happiness? About what here are they happy? Let's be honest, anyone who allows herself to step back and see will realize that expanding brackets of type is a somewhat tedious and joyless activity. Doing two or three or these things is fine. Doing more than that is utterly joyless. So what is this supposed happiness?)

But happiness – faux or otherwise - will often dissipate with presented with expressions of the ilk $(x+2)(x+6)(x+9)$ or $(x+y+2)(x+w+z+2+a)$. Where is FOIL now?

ON ANOTHER NOTE …

Suppose a student does find a means to be playful and creative in these little tasks and decides to LIFO instead? Is he wrong? Is he right but should be marked wrong because the system is "first, outer, inner, last" in that order, for the sake of uniformity in mathematics. Does the mathematics care in which order one expands brackets?

Working on understanding instead is more powerful and more joyful. Here is how I introduce the principle of expanding brackets to students. The mathematics doesn't care one whit about the order in which one expands brackets (nor does any mathematician!).


COMMENT: This essay is based on the material of chapter 1 of *THINKING MATHEMATICS!* **VOLUME 1.**

Let's begin with a geometric model of multiplication: "area." For instance, we would like to say that the multiplication of two numbers corresponds to the computation of the area of a rectangle.

e.g. Here's a 3-by-4 rectangle.

The computation $3 \times 4 = 12$ corresponds to dividing this rectangle into three groups of four unit squares giving a total of 12 unit squares.
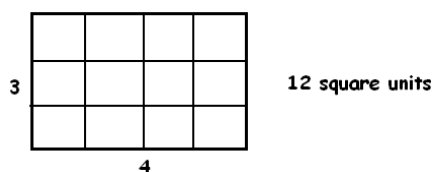
We can use this simple notion to our advantage.

EXAMPLE: Computing 23×37, that is, computing the area of a 23-by-37 rectangle, is a bit awkward to do directly.

However, one can simplify matters by breaking the "23" into two smaller numbers that are easier to work with, namely 20 + 3, and the "37" into 30 + 7. This corresponds to subdividing the rectangle into four pieces as shown:
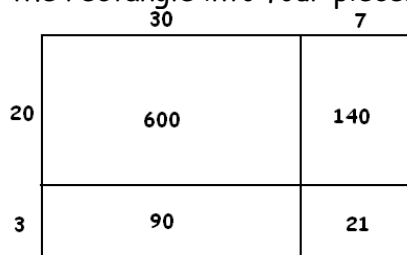
The area of the entire rectangle is just the sum of the area of these four pieces. We have:

$$23 \times 37 = 600 + 140 + 90 + 21 = 851$$

This is easy to compute! One can almost do it in one's head.

Notice what we did here. We wrote:

$$23 \times 37 = (20+3) \cdot (30+7)$$

$$= 20 \cdot 30 + 20 \cdot 7 + 3 \cdot 30 + 3 \cdot 7$$

This amounts to selecting one number from the first set of parentheses – "20" or "3" – and one number from the second set of parentheses – "30" or "7" – multiplying them together, and then adding all possible combinations of one number from each.

EXAMPLE: Compute 15×17.

Answer:

$$15 \times 17 = (10+5) \cdot (10+7)$$

$$= 10 \cdot 10 + 10 \cdot 7 + 5 \cdot 10 + 5 \cdot 7$$

$$= 100 + 70 + 50 + 35$$

$$= 255$$

Example: Compute 371×42

Answer: This corresponds to a rectangle subdivided into six pieces

$$371 \times 42 = (300+70+1) \cdot (40+2)$$

$$= 300 \cdot 40 + 70 \cdot 40 + 1 \cdot 40 + 300 \cdot 2 + 70 \cdot 2 + 1 \cdot 2$$
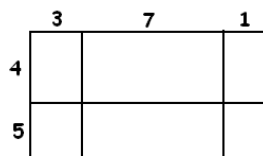
$$= 12000 + 2800 + 40 + 600 + 140 + 2$$

$$= 15582$$

NOTICE: Even for the expression $(300+70+1)\cdot(40+2)$ we simply select one term from each set of parentheses and make sure to cover all possible combinations.

EXAMPLE: The computation:

$$(4+5)\cdot(3+7+1)$$

corresponds to subdividing a rectangle into how many pieces?

Answer:

Six pieces!

We have:

$$
\begin{aligned}
(4+5)\cdot(3+7+1) &= 4\cdot3+4\cdot7+4\cdot1+5\cdot3+5\cdot7+5\cdot1 \\
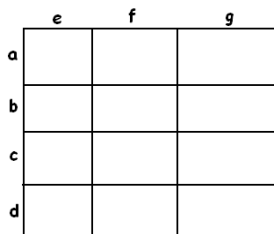&= 12+28+4+15+35+5 \\
&= 99
\end{aligned}
$$

☐

COMMENT: This was a very complicated way of computing $9\times11=99$!

EXAMPLE: What does this computation mean geometrically?

$$(a+b+c+d)\cdot(e+f+g)$$

Answer: This corresponds to subdividing a rectangle into 12 pieces.

With the appropriate patience, one could write this out:

$$(a+b+c+d)\cdot(e+f+g)=ae+af+ag+be+bf+bg+ce+cf+cg+de+df+dg$$ ☐

COMMENT: Mathematicians usually omit the multiplication sign "×" or "·" when multiplying two quantities represented by symbols.

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EXPANDING BRACKETS PRINCIPLE
To compute the product of two sums, select one term from each set of parentheses, multiply, and sum the results. Do this in any order you care. Just make sure to list all possible combinations.

e.g. $(a+b+c)(x+y+w+z) = ax+bx+cx+ay+...$

e.g. $(a+b) \times c = ac+bc$

e.g. $(x+y)(p+q) = xp+yp+xq+yq$

NOTICE: The second example above can be thought of the product of two sums with parentheses. Think of it as: $(a+b) \times (c)$. Selecting one term from each set of parentheses corresponds to always choosing "c" in the second set. It leads to the expansion: $ac+bc$.

COMMENT: This general principle speaks to an understanding of mathematics. FOIL doesn't. Plus it doesn't help for expressions of the sort $(x+y+x)(a+b+c+d)$, yet students who understand the subdivision of a rectangle and handle this with ease. So my advice to kids is to "just do it!" Expanding brackets by selecting one term from each set of parentheses in turn and multiplying is far better than memorizing a special rule that works only for one specific type of example.
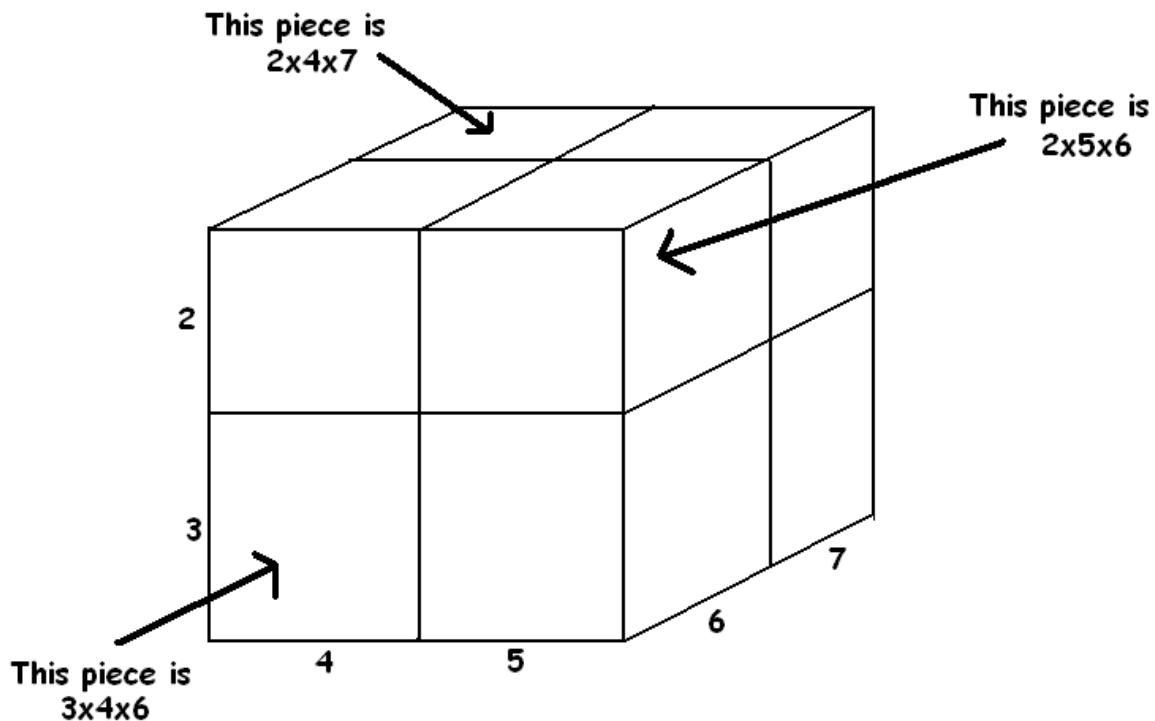
Now, let's get adventurous!

TAKING IT FURTHER …

If $(3+7)(4+5)$ corresponds to subdividing a rectangle into four pieces …

Question: What does $(2+3) \cdot (4+5) \cdot (6+7)$ correspond to geometrically?

Think about this before reading on.

Answer: $(2+3)\cdot(4+5)\cdot(6+7)$ corresponds to dividing a three-dimensional box into EIGHT pieces.

Here are the eight pieces:

$$(2+3)\cdot(4+5)\cdot(6+7) = \begin{aligned} & 2\times4\times6 \\ &+2\times4\times7 \\ &+3\times4\times6 \\ &+3\times4\times7 \\ &+2\times5\times6 \\ &+2\times5\times7 \\ &+3\times5\times6 \\ &+3\times5\times7 \end{aligned}$$

NOTICE AGAIN: This again corresponds to selecting one term from each set of parentheses and making sure to cover all possible combinations!

What we have actually shown in this example is that:

$$5\cdot9\cdot13 = 48+56+72+84+60+70+90+105$$
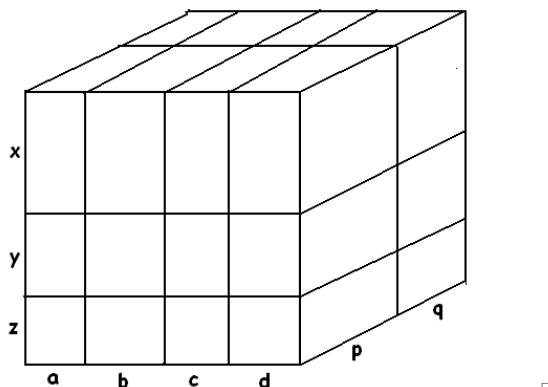
which is, indeed, correct. □

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EXAMPLE: If one were to expand:

$$(x+y+z)(a+b+c+d)(r+s)$$

how many terms would there be? What does this correspond to geometrically?

Answer: There would be $3 \times 4 \times 2 = 24$ terms. (Do you see why?)

The expression corresponds to subdividing a rectangular box into 24 pieces.

EXERCISE: If one were to expand:

$$(x+y)(x+a+b)(a+c+p)$$

   a) How many terms would there be?
   b) Would xac be one of those terms? How about cay, xcp, xax, and xyc?

EXERCISE: If you were to expand the brackets:

$$(a+b+c+d+e)(w+x)(a+x+b+t+r)(e+f)$$

how many terms would there be?

(Is there a geometric interpretation to this expression?)

COMMENT: When a number represented by a symbol x is multiplied by itself we write x^2 and say "x squared." Notice the word <u>square</u> here. We say this because of the geometry: x^2 is the area of a square of side-length x.

And what do we say for $x^3 = x \times x \times x$? Answer: x <u>cubed</u>. This is because x^3 is the precisely volume of a cube of side-length x.

Question: Why have we no word for $x^4 = x \times x \times x \times x$?

A WONDERFUL TRICK QUESTION:

Assuming we are comfortable with negative quantities …
(Read the appropriate essay on this topic).

What is the answer if one expands $(x-a)(x-b)(x-c)\cdots(x-y)(x-z)$?

HINT: Don't think too hard!

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FURTHER READING: *THINKING MATHEMATICS!* Volume 1.

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