TRANSCIPT OF TANTON'S STANDARD *EXPLODING DOTS* LECTURE

Here's a script of Tanton's standard close-to-one-hour Exploding Dots experience. Of course, this serves only as a guide and inspiration for you own approach to such a lecture. The key is to be your own human self, with your own personal human joy shining loud-and-clear through you to your audience.

THE SET-UP

I am often asked to give a 60-minute or 75-minute colloquium lecture with an audience sitting in an auditorium. This is not an interactive workshop experience. Nonetheless, I do insist

- I have access to a white board (top choice, quick and easy to erase dots with one's fingers), chalk board (second choice, not quite as easy to erase dots with one's fingers), or a document camera (third choice – can't erase dots with markers and paper, but one just makes do).

- each audience member has pencil and paper to try things out – even if it means balancing paper on his or her lap in a theater auditorium.

PRE- START

I am shameless. I always start with the INTERNATIONAL MATH SALUTE: http://www.jamestanton.com/?p=1328 (It is international now.) Whether or not I give away the solution right away or at the end of the session depends on my read of the audience.

THE LECTURE

[This part of the lecture matches **GMP Lesson 1]**

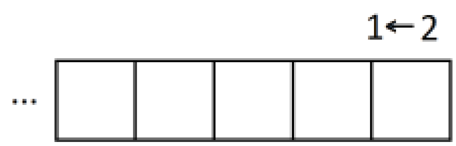
My banter is surprisingly consistent– and it always comes in an Australian/British accent mix. You will also see that I am deliberately playful with my language.

> *Let me tell you a story that isn't true.*
>
> *When I was a child, I invented a machine – not true. And all the machine was was a series of boxes that go as far to the left as I ever I please.*

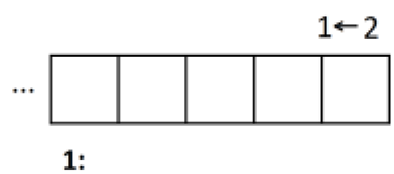
I draw a series of boxes on the board from right to left.

> *And being a precocious child, I gave this machine a name. I called it a "two-one machine" written in a funny backward way. (I was a child. I didn't know which way to write things!)*

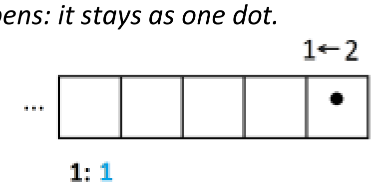
I label the boxes 1←2.



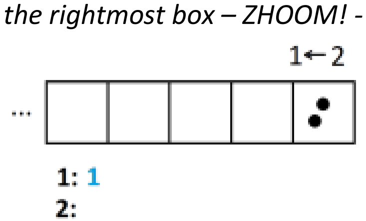
> *And the idea of this machine is that you put in dots. Dots always go in the rightmost box.*
>
> *Put in a first dot.*



> *And ZHOOM! Nothing happens: it stays as one dot.*

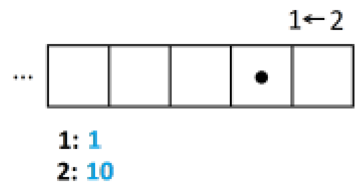


> *Put in a second dot – always the rightmost box – ZHOOM! - and then something happens.*



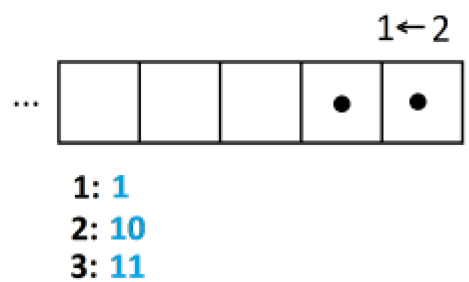
Whenever there are two dots in a box, they explode – KAPOW! <I erase the two dots> - *and are replaced with one dot one place to the left. Hence the name "1 ← 2 machine." Get it?*

The code for two in a two-one machine is nothing, nothing, nothing, one dot, no dot, which I'll just write as 10.



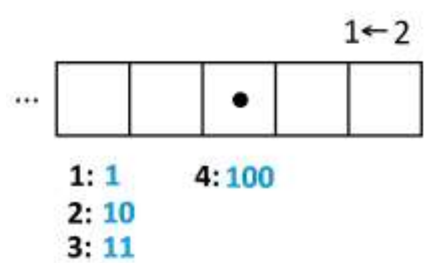
Comment: Although I happened to have drawn five boxes I don't write the code as 00010 with leading zeros. This is deliberate for two reasons: the number of leading zeros one draws should, technically, be infinite, and, we don't write leading zeros when writing numbers in base two or in any base. No one in an audience has ever questioned me on this, probably because I am telling the story in a way that it makes it clear it is "my story" and this is just how I set things up.

Here comes a third dot – ZHOOM! – the rightmost box. Nothing happens. The code for three is 11.



However, I think the code for four is going to be particularly exciting.

Draw in the fourth dot. Say KAPOW! Then say KABOOM! for the next explosion. We see that the code for four is 100.



Comment: By the way, as the lecture goes on I always use a different KA****** word for the explosion – kapow, kaboom, kazhing, kathwack, kathwoop, kabooble, kanick, ….

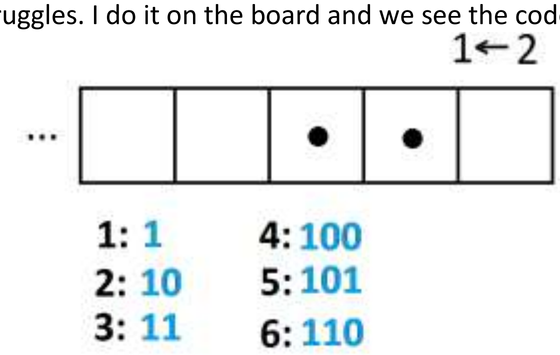
Can we see what the code for five is going to be?

The audience will say "101." They might say one-hundred-and-one and I warn that I am a bit nervous about saying hundreds and the like, as there certainly aren't a hundred dots anywhere here. Ask if we can just be safe and say one-zero-one?

Usually there is someone in the audience who can see this is binary and will yell out "It's binary" or "It's base two." I pretend I don't hear the comment.

The code for six?

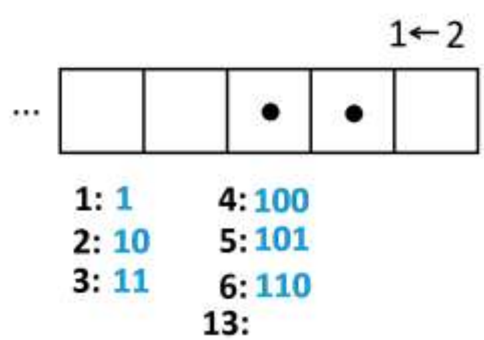
Sometimes the audience struggles. I do it on the board and we see the code for six.



Now I do a joke on the audience.

And the code for 13?

I write 13 on the board. The audience instinctively chimes 111, the code for seven. They catch themselves.



Working out the code for 13 can be tricky. There is usually someone who knows what is going on and will yell out 1101 right away. I repeat back 1101 in a questioning way. Usually then, someone yells out some other answer. I repeat that. Then I start saying all sorts of 1s and 0s in some confused manner.

111001? 10111..1? 11000010010100100111?

Depending on the audience we either let this question hang, or we solve it then and there. Many audience members will build their way up to 13 by working out the code for 7, and then 8, and then 9, and so on. I suggest an alternative method if we decide to solve it then and there.

Let's just throw 13 dots into the machine.

I clear the machine and draw thirteen dots in the rightmost box.

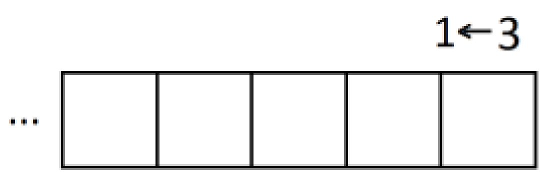
Will there be any explosions? How many?

I perform the six explosions and point out the one dot behind. The audience usually then suggests three more explosions, which we do, then one more. The code 1101 appears.

Hours of fun to be had for working out the different codes of numbers in my two-one machine.

But then, one day, in this untrue story, I had an epiphany. Instead of doing a two-one machine, I realised I could do a three-one machine.

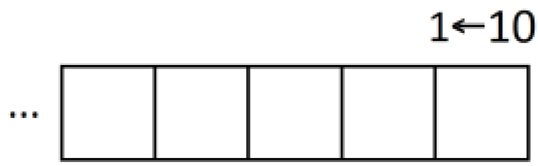
Don't erase any of the previous diagrams. Leave the 1←2 work on the board and go to a different area to draw a row of boxes for a 1←3 machine.



How does this machine work? Three dots in a box explode to be replaced by one dot, one place to the left.

I go through the same work as for the 1←2 machine, work out the codes of 1, 2, 3, 4, 5, and 6 and then do the same joke as before writing 13 next. We get/guess the code for thirteen is 111.

And MORE hours of fun to be had with a three-one machine. But then, one day, I had another epiphany. Instead of a three-one machine I realized I could do a … four-one machine …. or a five-one machine … Or a six-one machine. In fact, let's be wild, and go all the way up to a ten-one machine.



And let's be totally crazy. Let's put 273 dots into the ten-one machine.

I draw this machine on a third part of the board – preserving both the 1←2 and 1←3 work.

For the 1←10 machine, I start drawing dots in the rightmost box, first drawing them deliberately and slowly, counting as I go along, and then start drawing dots messily and crazily at speed.

Tell me when I have drawn enough dots.

The audience usually tells me to just stop. If not, I keep being really boring until they do.

By the way, someone invariably yells out that the code for 273 is going to be 273. I usually say something like …

Okay. I am slower than you. Remember I am small child in this story. Let me think this through.

First off … will there be any explosions? That's a yes/no question, by the way.

The audience will say "yes."

Initially, how many explosion?

"27"

Any dots left behind?

"Three."

I then mimic 27 explosions, erasing my mess of dots and saying lots of ka**** words. I then draw three dots in the rightmost box and write the number 27 in the second box.

Is it okay if I just write the number 27 here rather than draw twenty-seven dots?

"Yes."

Okay. Any more explosions?

"Yes. Two more."

Any dots left behind?

"Yes. Seven."

I make two more explosion sounds, erasing the "27" and then draw seven dots in the second box, and two dots in the next box over.

Yes. You were right. The code for 273 was indeed 2-7-3.

All right. I know you are on to me. What is going on? What are these machines really doing?

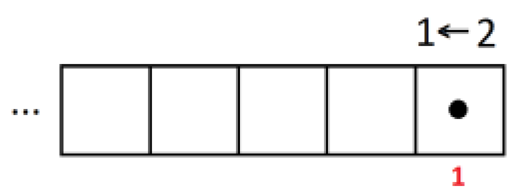
Invariably someone will yell out "bases" or "place-value."

Okay, but I really am slow. Let me take a moment to work my way through this.

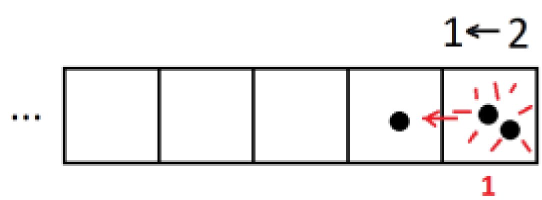
Let's go back to the two-one machine.

I walk back over to the 1←2 picture.

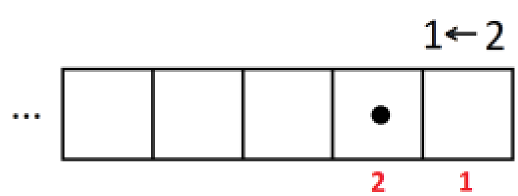
I set up this game so that dots in the rightmost box are always worth 1.



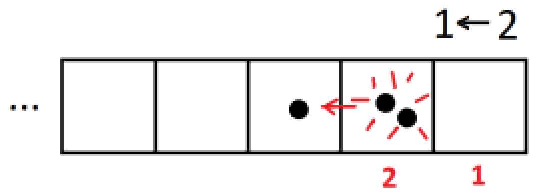
And we said that two dots here – two ones – is equivalent to one dot one place to the left.



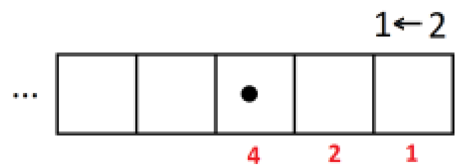
So a dot here must be worth … two.



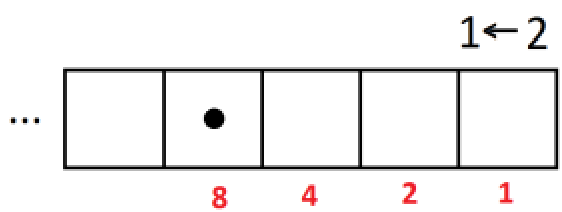
And by the same token, two dots here are equivalent to one dot, one place to the left.



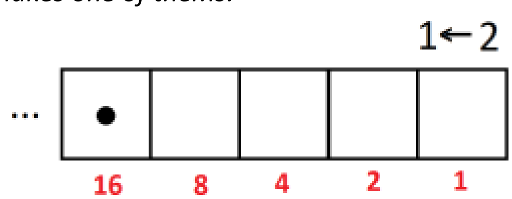
Thus a dot here must be worth two twos: four.



And two of these doth one of these make: eight.



And two of these makes one of thems.

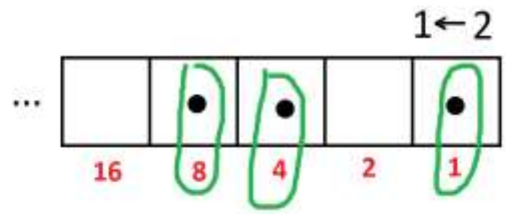


And two of thems makes one of theses: 32. And so on.

Sometimes an audience likes to keep listing powers of two. I do this, writing a scrawly list of powers of two up the left side of the diagram and do it until we decide that it is boring, or too hard, to continue.

Now, as a check, you told me that 13 was 1-1-0-1. Can you see now that you were right?

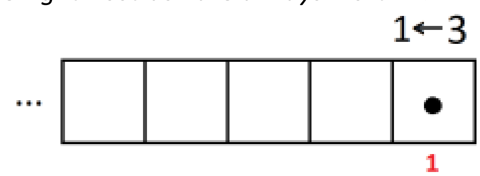
I draw one dot, one dot, zero dots, one dot back in the diagram and circle loops as shown to make it apparent we are seeing 13 as one 8, one 4, no 2, and one 1.



Comment: I have been asked, by teachers, in post-analysis of my lecture why I don't introduce exponential notation here. The simple reason is that it is not needed right now, it comes off as a bit forced, or at the very least, "school like" to want to fuss about this now. It is a diversion from the story. (However, this could serve as a moment for middle-school teachers to come back to and connect this story with an element of the curriculum.)

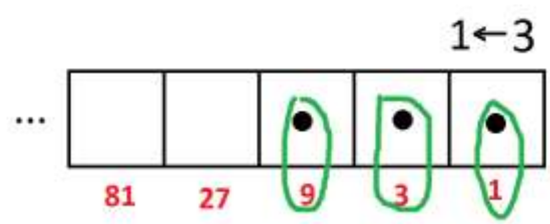
Now go to the 1←3 machine.

And here, dots in the rightmost box are always worth 1.

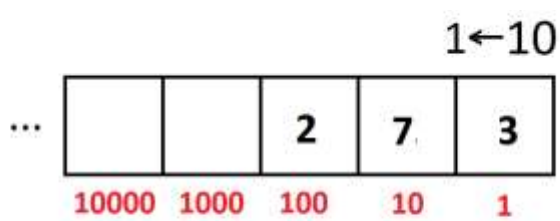


I go through similar banter to see the powers of three. (By the way, the audience will often say that three threes is six.) We get 1, 3, 9, 27, 729, 2157, … as high until we are bored.

We check that the claim that the code for 13 as 111 is indeed correct.



And for the ten one machine we get 1, 10, ten tens 100, ten one-hundreds 1000, and so on.



273: 273

So we have discovered "place-value" or "number bases." In fact, we see that we "speak" the language of the ten-one machine. When we say 273 we literally say ….

I write out the words "two <u>hundred</u> and seven<u>ty</u> three" on the board.

… two hundreds, and seven tens – that "ty" is short for ten - and three.

I might make a comment/joke here about how in U.S. schools you are not meant to say the word "and," it should be "two hundred seventy three."

Why do you think we humans have a predilection for the ten-one machine? Why do we like the number ten for counting?

The audience usually shows me ten fingers on their hands.

Yes. It seems that we naturally associate counting with the number ten because of our physiology. Many historians do believe this is the reason why we humans have favored base ten.

Martians, I happen to know, have six fingers on each of two hands. What base do you think they might use in their society?

"Base twelve."

Yes. Oh … and there are were some cultures on this planet that used base twenty. Where is that coming from?

"Fingers and toes."

Actually, there are some vestiges of base twenty in Western culture. In the U.S., how does the Gettysburg address begin?

"Four score and seven years ago."

How many years ago is that, actually?

There is usually a little umming and ahhing.

A score is actually an old word for twenty years, and so "four score and seven years ago" is thus four twenties and seven, that is, 87 years ago. That's a little bit of base twenty.

How do you say 87 in French? (Does anyone speak French?)

"Quatre-vingt sept."

Word-for-word that is literally four twenties and seven. It is base twenty again.

Back to the two-one machine … Computers use base two or binary. (I heard someone say that word before. I was pretending then I didn't hear it.)

I do acknowledge that I did hear what someone yelled out earlier on.

Why do we use base two for computer science?

Well, computers are built on electrical switches that are either on or off. So it is best to have a notation for arithmetic that is built on two symbols: 1 and 0, on and off.

Oh. There is talk of building optic computers that will be based on polarized light. You have light either polarized in one plane, or in an orthogonal plane, or have no light at all. What base system would be best for optic computers?

"Base three."

Yes. Ternary. Base three. This three-one machine.

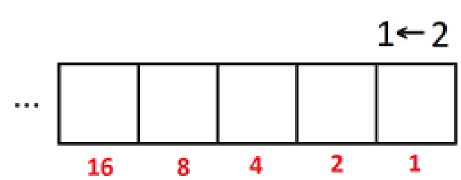
***

Okay, let's go back to the ten-one machine, the machine we use every day. Now that we've learned how to represent numbers in base ten (this is grade 1 or 2 of grade school, right?) let's now carry on and whip through the rest of the grade-school, middle school, high-school curriculum, and beyond!

But let's do focus on the ten-one machine.

I erase the board.

Comment: Many of my colleagues shorten this opening part of the lecture, especially at a hands-on workshop, and draw on the board or have printouts available right way, of boxes pre-labeled with base values, for the 1🡐2 , 1🡐3, and 1🡐10 machines. This way, participants can see within moments that, for the 1🡐2 machine, say, two counters in one box are worth one counter one place left.



[This part of the lecture matches **GMP Lesson 2]**

After you learn how to write numbers, what is the first thing you do with them?

Audience usually says "addition."

Okay, let's do, say, 251 + 124.

I write the sum on the board.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline \end{array}$$

We have 2 + 1 is 3 , 5+2 is 7, and 1+4 is 5.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline 375 \end{array}$$

Did you notice what I did there? I went left to right. Is that okay?

The audience usually tells me it is not.

You know … I always thought it was strange that in every class I was taught to read left to right EXCEPT in math class, when I was being told to do things right to left. That's weird!

What's the problem with going from left to right?

The audience usually mumbles something.

Oh! So the problem I did was too nice? You want me to do something like … 358 + 287?

I write the sum on the board.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

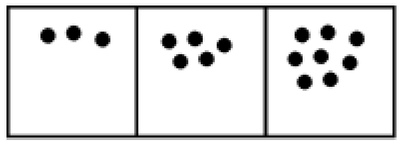
Well, 3+2 is 5; 5+8 is 13, and 8+7 is 15. The answer is five-hundred thirteenty fifteen!

I do the sum on the board still from left to right.

```
  3 5 8
+ 2 8 7
-------
 5|13|15
```

And I am right! Let me prove it to you.

Here's a picture of 358.



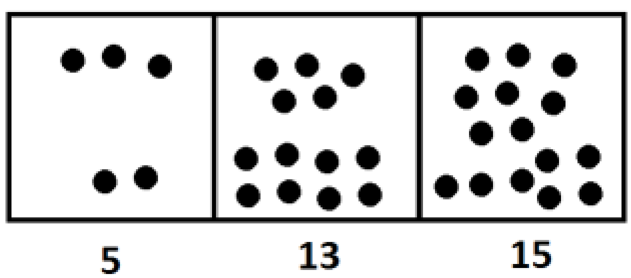
And I add to it two more hundreds – no worries.

I draw in an additional two dots in the hundreds place.

Eight more tens – you bet.

Seven more ones – not a problem.

And now I have a total of 5 hundreds, 13 tens, and 15 ones. "Three hundred and thirteenty fifteen."



This is an absolutely correct answer. There is nothing mathematically wrong with this answer. It is just that society thinks I am weird with I say three-hundred-thirteenty-fifteen.

So my question is: How can I fix up this answer for society's sake – not for mathematic's sake, but for society's sake.

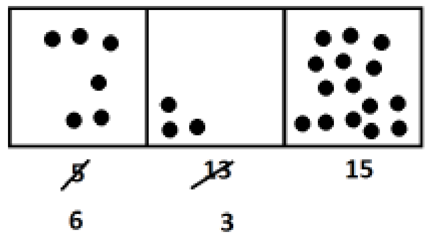
The audience will usually tell me to "carry."

What's "carry"? Remember this is a dots and boxes story and carry is not a word I have used. What's a "carry" in the language of this story?

"Explode!"

Okay. Which do you want to explode first – from the box of 13 or the box of 15?

I draw out the explosions of usually the middle box – just to keep gently breaking the mindset that math allegedly has to be right to left.

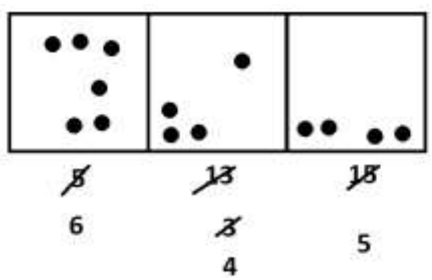


Six hundred and thirty fifteen.

Still a fine and lovely and mathematically sound answer. There are absolutely no problems with this answer.

"Explode again!"

But you are right. Society wants me to explode again.



Six hundred and forty five. That's an answer that society understands!

All right. So now I am curious. What is the standard algorithm for computing 358 + 287.

$$\begin{array}{r} 358 \\ +\ \underline{287} \end{array}$$

You start at the right <knowing look to the audience> *and go 8+7 equals 15. But you are not allowed to write 15, you just write a five and put a little one at the top of the middle column.*

$$\begin{array}{r} 1 \\ 358 \\ +\,287 \\ \hline 5 \end{array}$$

What just happened there?

"We exploding ten from the thirteen and added a dot to the tens place."

And then we go 1+5+8 is 14, just write 4 and "carry," that is explode ten to make a one next place over. And then 1+3+2 gives 6 and the answer 645 appears.

$$\begin{array}{r} 11 \\ 358 \\ +\,287 \\ \hline 45 \end{array} \qquad \begin{array}{r} 11 \\ 358 \\ +\,287 \\ \hline 645 \end{array}$$

So the standard algorithm works from right to left and does the explosions as you go along. I personally like to go from left to right, as I was taught to read, and do all my explosions at the end.

In fact, back in the days one used to have to balance one's checkbook by hand, I really did do it that way: I'd add up all the ones, tens, hundreds, thousands – well, there never were any thousands! – and just write the totals at the bottom. Then I would do all the explosions to get the final answer.

Actually, it is interesting that society doesn't let you say five-hundred-and-thirteenty-fifteen. It wants single digits in each place. Ohh! That's not quite true. I am allowed to say "twelve hundred and fifty nine."

1259

Isn't that weird? Apparently we are allowed to say twelve-hundreds. Hmmm.

Comment: If I sense there is time and the audience is inclined I ask if the audience wants to "*try one on their ownsies"*? I'll have them do something like 5794 + 2459, asking first for the James Tanton answer, and then to fix up that answer for the rest of society. (Note: I usually make sure the first two digits sum to less than 9 to avoid carries at the very front. It just obviates an extra knot of thought. But if the audience is up for that, we can do that too.)

[This part of the lecture matches **GMP Lesson 3]**

Okay. Addition done. What's the next operation students usually learn?

"Subtraction."

Too hard! Let's do multiplication.

Comment: I make this a speedy turnaround. It induces a laugh.

Okay. What's 26417 x 3?

26417 x 3

You've got three seconds to write down the answer. Just write it down. I'll give you three seconds to do it. 1, 2, and 3. Okay, what's the obvious and easy answer?

I've given the audience no time whatsoever to think. But someone invariably answers 6-18-12-3-21.

Yes. Here's a picture of 26417. Is it okay if I just write numbers rather than draw dots?

2	**6**	**4**	**1**	**7**

And we're saying to triple this number.

2	**6**	**4**	**1**	**7**	**x 3**

So right now we have 2 ten-thousands, and if we triple this, we'd have 6 ten-thousands.
Right now we have 6 thousands, and triple would make 18 thousands.
4 hundreds becomes 12 hundreds; 1 ten becomes 3 tens; and 7 ones, 21 ones.

6	**18**	**12**	**3**	**21**

So we do get the answer sixty eighteen thousand, twelve hundred and thirty twenty-one. Absolutely solid and mathematically correct!

Now, how can we fix up this answer for society?

"Do some explosions. "

Which do you want to explode first?

I usually follow someone who suggest exploding one of the middle boxes first, again just to break the regimented mindset.

Do you want to keep going or just say that we know what to do to finish this up for society?

Depending on what the audience wants, either we finish this on the board to get the answer 79251 or we just leave it be and move on.

Comment: This is all I do for multiplication. One error in the gdaymath.com materials is that I omitted a discussion on how to do long multiplication in this model. We will include long multiplication in our Global Math Project materials.

If asked about long multiplication, I say that it can be done and I invite the person to think about it themselves. As a suggestion, I might say that multiplying by 23, say, is the same as multiplying by 20 and by 3 and then combining the two results. Alternatively, we could just say that 143 x 23 is 23|92|69 and do long multiplication this way.

(By the way, in this approach an interesting question arises: Is it obvious that multiplication should be commutative? In this approach there doesn't seem to be any reason to suspect that multiplication should commute.

24 x 23 = 46|92
23 x 24 = 48|72

It is astounding all the explosions give the same final result in each case, and always does so in all such examples! What a lovely mystery for the class this invokes.)

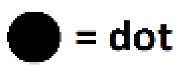
[This part of the lecture matches **GMP Lesson 4]**

Okay. Subtraction. You did say subtraction earlier on. Let's do it now.

Here's my problem subtraction: I don't think subtraction exists. To me, subtraction is just addition – it is the addition of the opposite.

So that means in my model of dots and boxes I am going to need the notion of the opposite of a dot.

Now here a dot is solid circle.



What do you want to call the opposite of a dot? And how should we draw one?

The audience invariably suggests the names "negative dot" and "opp dot" and "opposite dot" and "tod" (get it?) and "anti-dot." They suggest I draw Xs or squares or open circles.

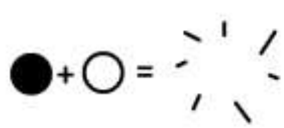
I say I'll go with open circles because they are quick and easy for me to draw on the board and I worry about Xs looking too much like symbols already being used in algebra class. I also say that I like the name "anti-dot" because it reminds me of matter and anti-matter. (But I will sometimes go with "tod" if the person who thought of it seems particularly pleased with the idea.)

What happens if you bring matter and anti-matter together?

"They explode!"

Well … we've already got explosions in our story. How about we say that they just go "POOF!" and annihilate? Bring a dot and antidot together and they go POOF!

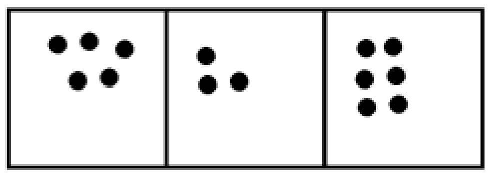
My "poof!" sound is soft and breathy.



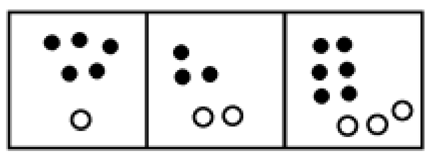
All right. Let's do, 536 – 123.

$$\begin{array}{r} 536 \\ -\ \underline{123} \end{array}$$

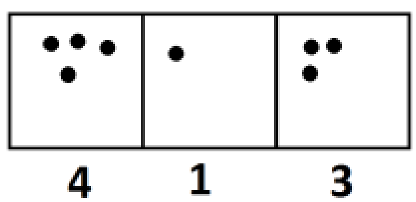
Now 536 is five dots, three dots, six dots.



To this we are adding the opposite of 123. That's one anti-hundred, two anti-tens, and three anti-ones.



Any annihilations? Yes – lots of them. POOF!   POOF POOF!   POOF POOF POOF!



We see the answer 413 appear.

It is though as I went from left to right and just said "5 take away 1 is 4", "3 take away 2 is 1," and "6 take away 3 is 3." Left to right again!

$$\begin{array}{r} 536 \\ -\ \underline{123} \\ 413 \end{array}$$

All right. I know what you are thinking. This example is too nice! Let's do something more juicy. How about … 512 – 347?

512
-347

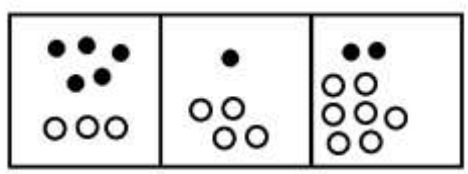
Well, going from left to right, "5 take away 3 is 2, 1 take away 4 is -3, and 2 take away 7 is -5." The answer is two-hundred negative-three-ty negative-five.

512
-347
2|-3|-5

This answer is absolutely mathematically correct. I'll prove it to you!

I draw out the dots and antidots.

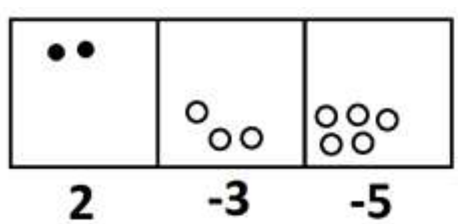
Five hundreds and one ten and two ones, plus three antihundreds, four anti-tens, and seven anti-ones gives …



And after annihilations

Lots of POOF! Sounds

We have two actual hundreds, three anti-tens, and five anti-ones: two-hundred negative-three-ty negative-five.

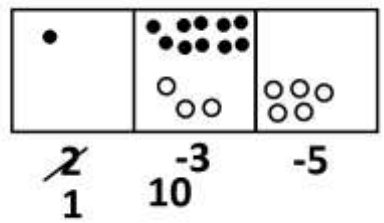


Okay. How can we fix up this answer for the rest of the world?

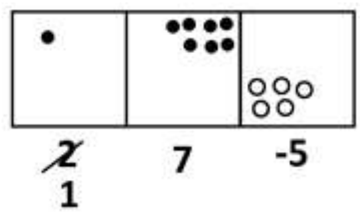
Invariably someone suggests “unexplode.”

Brilliant! One of these dots must have come from ten dots to its right, so let’s unexplode it to bring back those ten dots.

I make an air-being-sucked sound for an unexplosion.

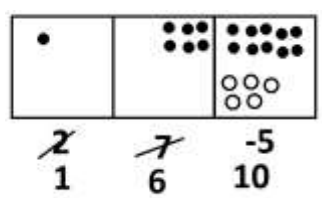


Some POOFS for annihilation.

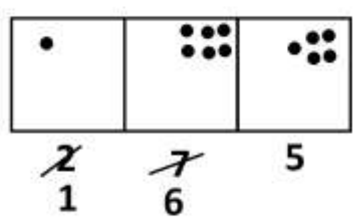


One-hundred-and-seventy-negative-five. Beautiful!

Shall we unexplode again?



More POOFS.



There it is.

So how does this compare with the traditional algorithm? 512 take away 347.

$$\begin{array}{r} 5\,1\,2 \\ -3\,4\,7 \\ \hline \end{array}$$

You start at the right and look at "2 take away 7" and say it can't be done.
So what do you do?

"You borrow a one."

Okay, so you take a dot from the tens column and unexplode it to make ten ones. That leaves zero dots in the tens column. And I suppose we should write ten ones to go with the two in the ones column.

$$\begin{array}{r} \,0\,{}^{10} \\ 5\,\cancel{1}\,2 \\ -3\,4\,7 \\ \hline \end{array}$$

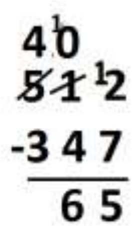
But we don't. We put a 1 in front of the two and make it twelve. (Oh! That is 10 and 2 !)

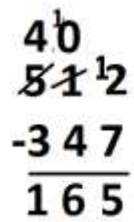
$$\begin{array}{r} \,0\, \\ 5\,\cancel{1}\,{}^{1}2 \\ -3\,4\,7 \\ \hline \end{array}$$

Then we say "twelve take seven is five."

$$\begin{array}{r} \,0\, \\ 5\,\cancel{1}\,{}^{1}2 \\ -3\,4\,7 \\ \hline 5 \end{array}$$

Next: "Zero take away four" Can't be done. So you unexplode – that is, carry – again, to make it 10 take away 4 to get 6. And then 4 take away 3 is 1.





Again. It is a style thing. The traditional algorithm has you work from right to left and do all the unexplosions as you go along. I prefer to JUST DO IT! Work from left to right and do all the unexplosions at the end.

By the way, let me point share with you what a young student pointed out to me when I showed her my approach. Here's two-hundred-and-negative-thirty-negative five again.

5 1 2
-3 4 7
2|-3|-5

This is really two hundred, negative thirty, and negative five!

5 1 2
-3 4 7
2|-3|-5

200
-30
-5

We see the answer 165.

If time, I might ask the audience if they want to do one on their "ownsies." I suggest something like, 487616 – 199257 and ask, first for the James Tanton answer, then to fix up that answer for the rest of the world.

[This part of the lecture matches **GMP Lesson 5]**

Addition, subtraction, multiplication. Next … division.

Now we come to the true part of today's story.

I remember being taught long division back in Australia in grade 5. Ms. Delaney's class. And here is how I was taught to do something like $276 \div 12$. *It was a really bizarre method. Truly bizarre.*

In fact I remember not understanding the method at all, because it was so bizarre. And I remember I was working solely to please my teacher. In fact, I was absolutely cognizant that I did not have a clue what I was actually doing, that I was just performing. And I was one of the top students in the class getting gold stars on my long division worksheets, but I knew I didn't understand what I was doing one whit!

Comment: I might pause at this moment to build up the suspense of matters a wee bit and ask a tangential question. *By the way … does anyone know the name of this symbol:* $\div$*? Yes. It's the division symbol. But it has an official name, a Latin name. Do you know what it is? It begins with an O? OB? OBE? It's an "OBELUS." I say, if we are going to have kids memorize jargon, let's have them memorize delightfully quirky jargon.*

Okay. $276 \div 12$. *First we had to draw a funny curved symbol with a line. Then we had to write the big number – the quotient or quotior or quotiee, I really don't know the names for these things - under the symbol and the smaller number to the left. (Is this like something you had to do?)*

$$12\,\overline{)\,276}$$

Now remember how everything in math is right to left? Well, not here. Of course not! Why would you think so? So look at the 2-7-6 and focus on just the 2. (Why "two", I thought. The number is 276, not two. This is weird.) Ask how many times does 12 go into 2. The answer is that it doesn't, so don't be silly.

So don't look at 2, look at 2 and 7, that is, twenty-seven. (Now this really bizarre – the number is not twenty-seven, it is 276. Why are we looking at twenty-seven?)

Anyhow, it matters not. You just go ahead and ask: How many times does 12 go into 27? Two times.

We have to then write 2 at the top of the diagram –and this was the really scary part because the alignment apparently really mattered. I think the 2 had to go in the middle position.

$$\begin{array}{r} 2 \\ 12\overline{)276} \end{array}$$

Next, back in Australia, we have to write 2 x 12 = 24, and then suddenly do a little subtraction problem in the middle of the division problem. Who knows why?

$$\begin{array}{r} 2 \\ 12\overline{)276} \\ \underline{24} \\ 3 \end{array}$$

27 take away 24 is 3.

Now if all that isn't bizarre enough now comes that most bizarre part of all. We got three. There's a six. Back in Australia we had to draw an arrow, bring the six down, and that three just magically changes to 36! It just does!

$$\begin{array}{r} 2 \\ 12\overline{)276} \\ \underline{24}\!\downarrow \\ 36 \end{array}$$

Now you ask: how many times does 12 go into 36? Three times. You write three at the top, do another subtraction problem and get zero, which apparently is good. And we have the answer 23.

$$\begin{array}{r} 23 \\ 12\overline{)276} \\ \underline{24}\!\downarrow \\ 36 \\ \underline{36} \\ 0 \ \checkmark \end{array}$$

Bizarre indeed!

Leave this algorithm on the board.

So can we help out young James Tanton? Can we make sense of what is going on here?

First of all. Are we clear on what division really is? What do we mean by $276 \div 12$ *is 23?*

People in the U.S. seem to cry out "it means repeated subtraction," which feels just slightly odd to me, so I rectify it by saying …

Right, it means pulling out groups of 12. That is, if I have a collection of 276 dots …

I start drawing a load of dots on the board, a big mess that we presume represents 276 dots

… and start identifying groups of 12 within it …

I draw loops circling, supposedly, groups of 12.

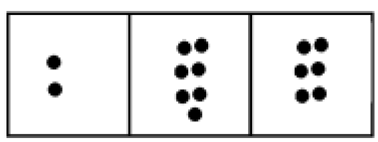
… we'd see 23 groups of 12.

So $276 \div 12$ *is asking: How many groups of 12 do you see in a picture of 276.*

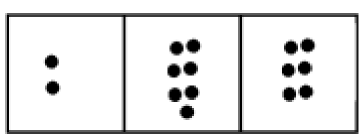
Okay. But drawing a big messy picture of 276 dots like this is an extraordinarily inefficient way to do this.

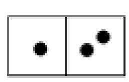
Comment: What I am about to do assumes I have an audience very familiar with upper-school and middle-school mathematics. If I have an audience of very early grade teachers I do $3906 \div 3$ here just as a segue to understanding the dots-and-boxes picture of $276 \div 12$ coming next. See the notes at the end of this section for how I do the $3906 \div 3$ example at this point.

So let's draw 276 in our dots and boxes scheme. It looks like this.

276 = 

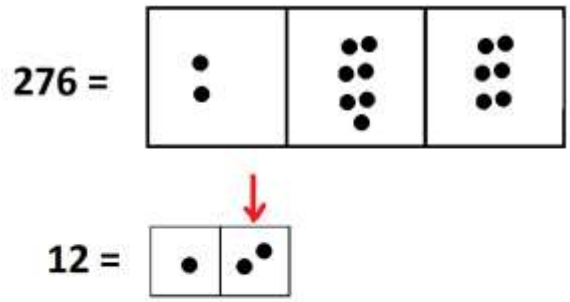
And we're looking for groups of 12 in this picture. Here's what 12 looks like.

276 = 

12 = 

Ooh. Here's something to note. I didn't actually physically draw 12 dots just then. So where are the 12 dots really?

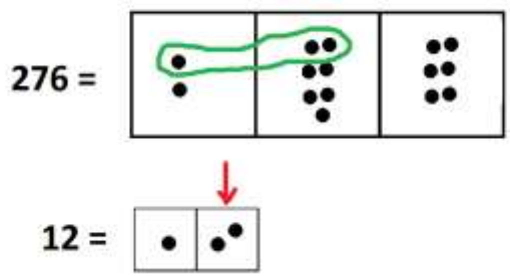
Well, that single dot to the left must have come from an explosion. So all 12 dots are really in the rightmost box – it's just that an explosion occurred.



Okay. So we're looking for pictures of 12 in the picture of 276. Do you see any one-dot-next-to-two-dots anywhere in the diagram?

The audience will likely yell out several spots.

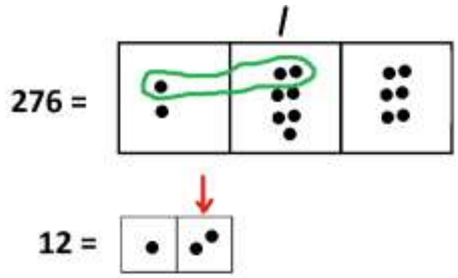
Okay. Here's one-dot-next-to-two-dots.



Now, within the loop. Where are the 12 dots really? In the left part of the loop or the right part of the loop?

"The right part of the loop."

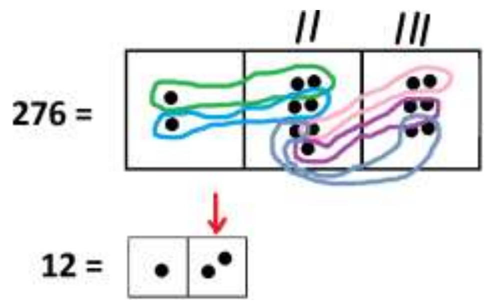
Okay. So we have one group of 12 sitting right here at the tens position.



Any more?

"Yep."

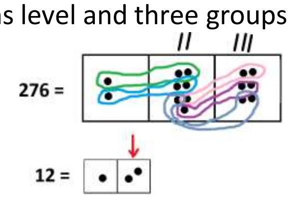
Another one at the tens level.
And one at the ones level. And another two at the ones level.



So we see two 12s at the tens level and three 12s at the ones level. We see the answer 23.

This usually induces the reaction "Whoa!"

Comment: This picture is deceptively subtle, but folk tend not to question it in a lecture, and I let the subtlety just slide. But here's what we actually just accomplished. Our picture shows that 276 can be seen as two groups of 12 dots at the tens level and three groups of 12 dots at the ones level.



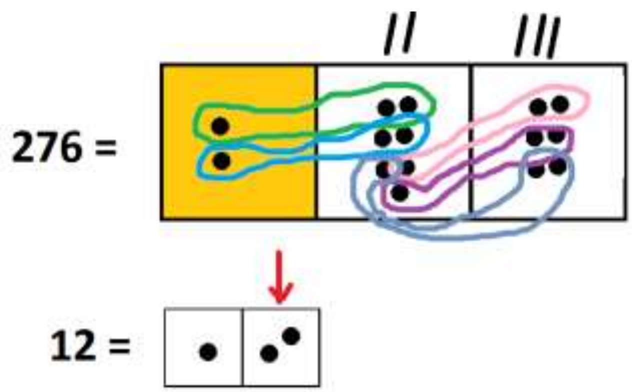
In dividing everything by 12, each group of twelve dots would be replaced by a single dot. This would then yield two dots in the tens level and three dots in the ones level. That is, the groups of 12 we have identified indicate the answer to the division problem of dividing by 12. The answer is 23.

(If you like, we can say that the picture shows we can rewrite 276 as $2\times120+3\times12$. Dividing by twelve thus gives $2\times10+3\times1$.)

Can we go back and help young James Tanton. What is the traditional algorithm doing?

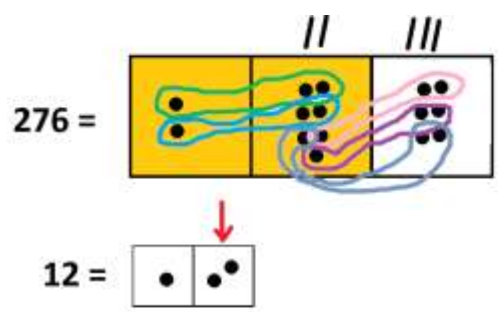
Go back to standard algorithm that was left on the board.

When I was told to just look at the number “2” in 276, what I am looking at in the dots and boxes picture? Just the two dots in the leftmost box.

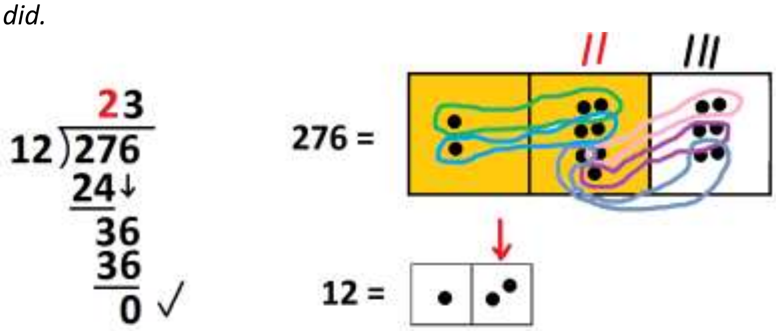


And do we see any 12s in just that part of the picture? No. Of course not.

So what’s this focus on “27” in 276? It means we’re now focusing on the first two boxes of dots.

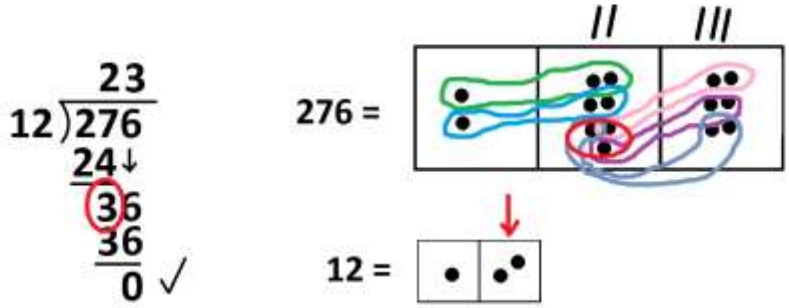


Do we see any 12s here? YES! We saw two. And we marked 2 at the tens position, just like the standard algorithm did.



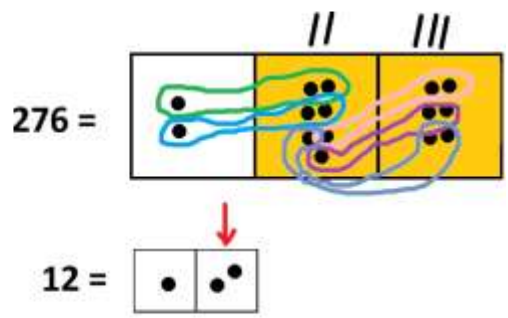
Now what is all this subtraction business?

Well, it says that we've dealt with two groups of 12. We might as well take them out of our consideration. But there 3 dots left over still to be dealt with.

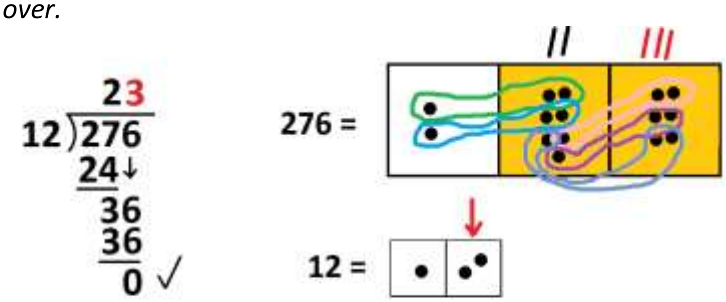


Now, this "bring down the 6 and magically change 3 to 36." What is that all about?

Well, it says just shift your attention to the next two boxes of the picture. We have the 3 dots left over in the middle and the 6 dots to the right.



We now see three 12s, we tally 3 in the ones position, and a final subtraction confirms that there are no more dots left over.



So the traditional algorithm is doing the right thing after all – and, of course, was – that's why we've been teaching it for centuries. But now, finally, it makes sense to me. Thank you!

Comment: Sometimes an audience member will ask about remainders. One can address this quickly right now, by asking about $279 \div 12$: *How will the picture change?* You will see three dots left over in the picture, and so $279 \div 12 = 23\ R\ 3$ or $23 + \frac{3}{12}$ (twenty-three plus three dots still to be divided by twelve.) *So if there are remainders, you just see them. And if there are no extra dots floating about the place, then you know you have an evenly divisible problem.* Erase the three extra dots and change 279 back to 276. Restore the $276 \div 12$ picture.

Now, do not, under any circumstances, erase the $276 \div 12$ picture from the board. Make it seem quite casual that it you happen not to erase that picture at any stage of the next part of the lecture. This picture is key to a mind-blowing epiphany coming up later.

Do you want to try one your own? Of course, the answer is YES!

Try $2783 \div 23$ *the dots and boxes way.*

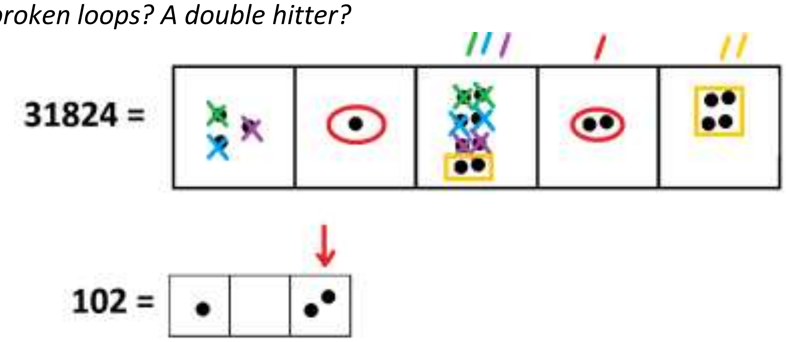
Have participants do it and see the answer 121. Do the problem on the board too.

Try $31824 \div 102$. *This one has a little difficulty to contend with.*

Have participants try it. The number 102 is represented as one-dot-blank-two-dots, spread out over three boxes. Some audience members will be circling one-dot-two-dots, the number twelve.

Show the answer on board.

Did you find drawing loops annoying? Yeah. I agree. What if I just did little Xs instead? Is that easier? Some broken loops? A double hitter?



Comment: If this is a longer lecture, I might have participants also try $3900 \div 12$ (requires unexploding) and $2798 \div 11$ (has remainder) and $4366 \div 14$ (which has both).

IMPORTANT COMMENT FOR AN AUDIENCE OF VERY EARLY GRADE TEACHERS

The presentation I take for $276 \div 12$ is often too swift for folk not fully immersed in the upper-grade school curriculum, and so, for such an audience, I usually draw and examine the dots-and-boxes picture for $3906 \div 3$ before doing the dots-and-boxes analysis of $276 \div 12$.

And just to make sure we are fully comfortable with division, let's do an example that we can see the answer to right away. What's $3906 \div 3$?

"1302"

So let's think about how we know that so quickly.

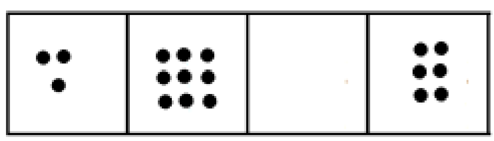
In asking for $3906 \div 3$, *we're asking for how many groups of three can we find in a picture of* 3906 *dots drawn on a page. Drawing this out is the inefficient approach.*

But we do notice that 3906 is 3000, and 900, and 6. Dividing by three we get 1000 and 300 and 2, that is, 1302. That's what we did in our brains just now.

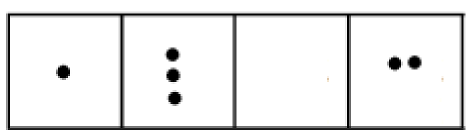
$$3906 = 3000 + 900 + 6$$
$$3906 \div 3 = 1000 + 300 + 2$$

Can we see this in our dots-and-boxes picture?

Here's what 3906 looks like. We literally see 3 thousands, 9 hundreds, and 6 ones.

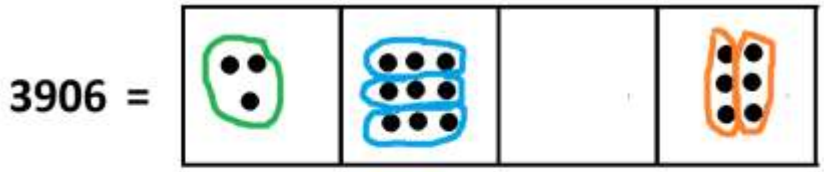
3906 =


Dividing by three gives this picture.

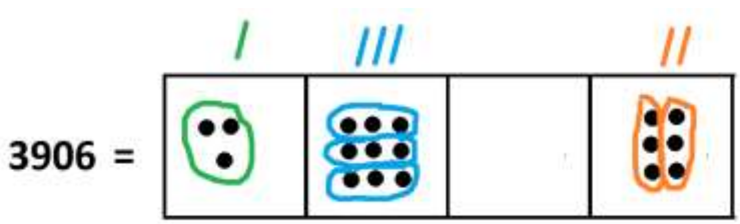
1302 =


What happened here? For each group of three dots we could identify, we drew one dot. After all, upon division by three, three dots become one dot.

Back in the original picture here are the groups of three we can see.



And when we do the division we'll see 1 dot in the thousands position, 3 at the hundreds position, and 2 at the ones position. But rather than ruin the picture I have now, let me just indicated that with tally marks at the top.



Now I can see that identify groups of three tells me what I am going to get when I divide by three.

If time, do the picture of $426 \div 2$ for a quick practice check,
do $402 \div 3$ to introduce unexplosions,
have participants practice with $61230 \div 5$.

Okay. Now let's go back to grade five James Tanton and help him out with $276 \div 12$.

[This part of the lecture matches **GMP Lesson 6]**

Okay. I've gotten us through basically all of grade school. Let's know whip right on up to advanced high school algebra!

Here's the thing about everything we've been doing: none of it is locked into being a ten-one machine. We could do all this work in a two-one machine, or a five-one machine or a ninety one-one machine. The math doesn't care which machine we do it in. It is only us humans with a predilection for the number ten that are gung-ho for the ten-one machine.

What I want to do now is repeat this entire, but now in all possible machines all at once!

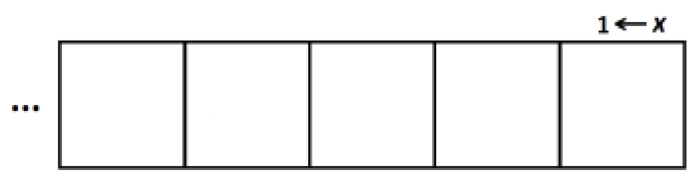
Let me explain what I mean by that.

Well, let's do it all again in a machine, but this time I am not going to tell you what machine we are in: it could be a ten-one machine – I am just not telling – or it could be a two-one machine, or a seven-one machine. I am just not going to tell you. It is the mood I am in.

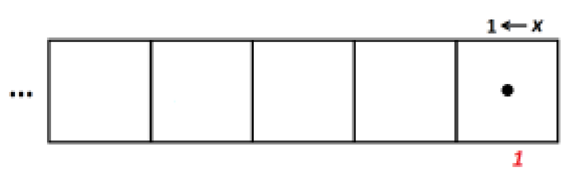
Now … just to give a nod to high-school algebra … there seems to be some favorite letter of the alphabet that algebra teachers use over and over again if you've got a number but just don't know what number it actually is.

"x"

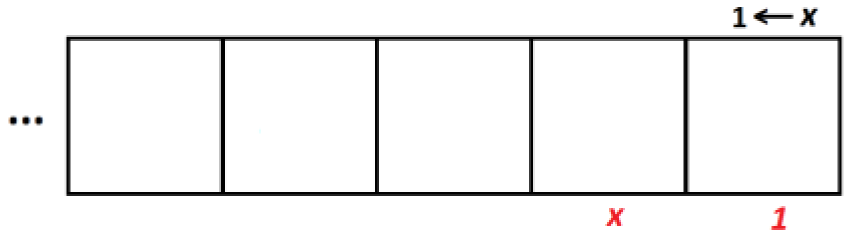
Yes. So let's look at an x-one machine. I am just not going to tell you what x is in my head. It could be ten, for a ten-one machine. But I am not saying.



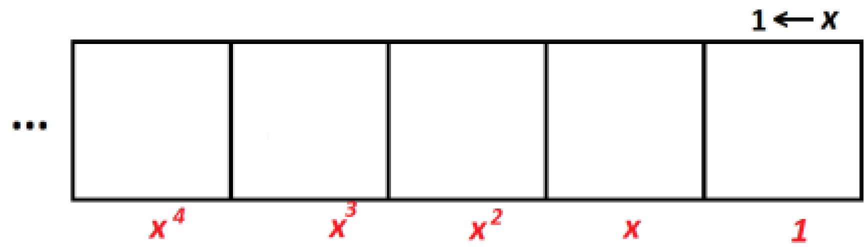
Now we do still know that dots in the rightmost box are always worth 1.



And that the next dot over is worth x of these, x ones, which is x.



And x xs makes one of these: x^2. And x x^2s makes one of these: x^3, and so on.



Now, as a check, suppose I told you x really was 10 in my mind. Then we'd have 1, 10, 10^2 = 100, 10^3 = 1000, and so on. This really is a ten-one machine. Or if I told you x was 2 in my mind, then we'd have 1, 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, and so on. This is a two-one machine.

Okay, so an x-one machine really does represent all machines all at once.

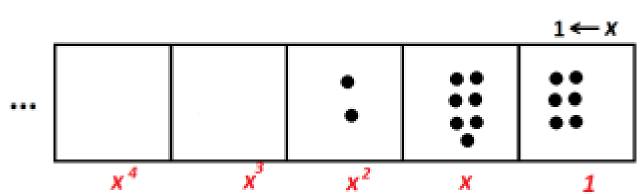
Now, we've just done a whole load of grade 5 arithmetic. Now, out of the blue, I want you to do this problem from advanced algebra.

$$2x^2 + 7x + 6 \ \div \ x + 2$$

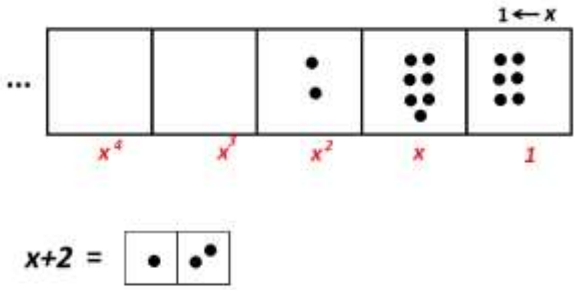
I am serious. Do it! (And yes, I know high school teachers are yelling at me right now: I should put parentheses throughout all this. But you know what I mean!)

The audience usually balks at how visually scary this is. I cajole and comfort, but still insist they do it. It doesn't take long for people to get into, and then I do it on the board with them.

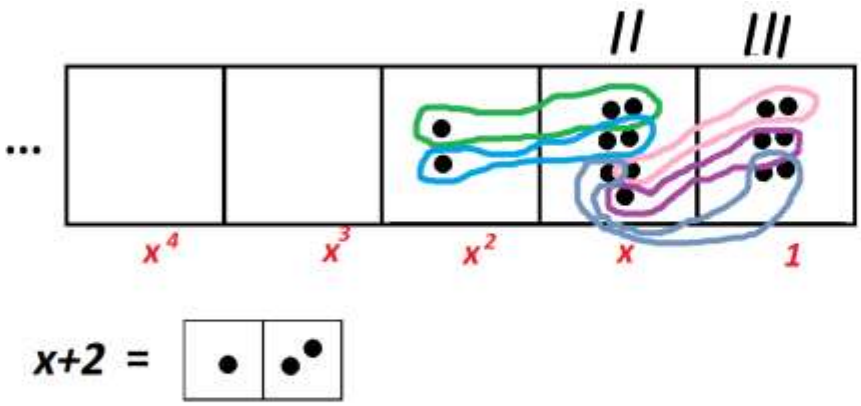
What does 2x^2 + 7x + 6 look like?



And what does x+2 look like?



Now, do we see any one-dot next to two-dots? Loads of them!



So how do we read this answer?

"2x plus 3"

Yes!

$$2x^2 + 7x + 6 \div x + 2 = 2x + 3$$

Tell me. Did that just feel like deja-vu?

I go over to the early part of the board, lean against the picture of $276 \div 12$, and just happen to rap my fingers on the board right by it.

Do you see that we actually did all this before? Look at the two pictures. We have an identical picture to $276 \div 12$!

"WHOA!"

In fact, suppose I tell you that x really was 10 in my head all along.

I wander back over to $2x^2+7x+6 \div x+2 = 2x+3$.

> *What number is* $2x^2+7x+6$ *? It's two-hundred and seven-ty six. Divided by … twelve … gives the answer … twenty-three. Just as we did over there!*

Comment: Depending on my feel of the audience I might add: *If this was a two-one machine instead, that is, x is two, what division problem did we do? "2 times four + 7 times two + 6 ones", that is, 28, divided by "two + 2," that is, 4, equals, "2 times 2 + 3," which is seven. And that is correct. We've really done a whole infinity of different division problems all in one hit!*

> *So this is it! This is all there is to high-school polynomial algebra. It is just a repeat of grade 5 arithmetic, but in base x rather than base 10. It is no different! And I am not sure why people tend not to point this out.*

> *Do you want to do one on your ownsies?*

"YES!"

> *Okay. If you can do this one, you can do any one! Try …*

$$\frac{x^4+2x^3+4x^2+6x+3}{x^2+3}$$

Comment: Depending on how snarky I feel (which is often the case) I might say two things here.

1. *So much curricula are obsessed with making school mathematics "real world," connecting everything to real world problems. Would you like me give you a real world application of this division problem?* "YES!" *Okay. I was in the grocery store the other day and they wouldn't let me check out until I solved this polynomial division problem. There. That's my real world example.*

2. *Do you have to do something called "synthetic division"?* "Yes" *Forget it! It's too hard and weird. Just divide the two polynomials! In fact, synthetic division is useless for you in this example. You can only divide by linear terms with synthetic division. Here we are dividing by a quadratic.*

The audience invariably figures out the correct answer. It is x^2+2x+1 . I do it on the board to confirm.

Comment: This can be a satisfying place to stop if time is up or if the audience is not comfortable with high school mathematics. Seeing this connection to advanced algebra, and feeling it is within reach, is very satisfying.

[This part of the lecture matches **GMP Lesson 7]**

Okay. Now that we are feeling really good about doing advanced algebra, I have a confession to make. I've been pulling the wool over your eyes. Everything I've been saying is a lie!

Well, that's not quite right. I've been choosing examples that are actually too nice and just work out beautifully. The truth is, this fabulous method of ours actually doesn't usually work.

Consider, for example,

$$\frac{x^3-3x+2}{x+2}\ .$$

Do you see what I've been avoiding all this time?

"Negative numbers!"

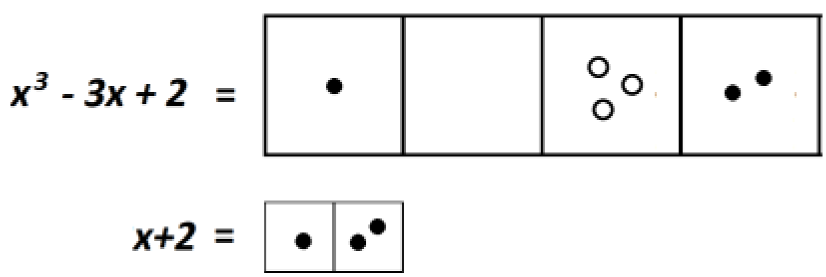
Yeah.

What happens if you try to draw this one? Can you draw it?

"You have to use anti-dots."

Okay.

I draw it on the board following the audience's lead.



So ... Do you see any one-dot-next-to-two-dots in this picture? I don't.

I want to weep.

Like I said … the method is useless. There is nothing to do.

At this point someone in the audience invariably suggests unexploding a dot, specifically the one on the far left.

> *Brilliant idea! How many dots do I draw?*

"Ten dots"

At this point I usually stay silent and still and stare at the audience. They realize that we can't assume this is a ten-one machine. In fact, we don't know what machine we are in – we don't know how many dots to draw.

> *Oh bother!*
>
> *That was a great idea, but it proved not to be useful.*

Comment: Someone might suggest drawing *x* dots. My response is: *I am not sure how to draw that, but if you can think of a way to make that idea work, I say, go for it!*

> *So unexploding proved not to be helpful. Like I say, I just want to weep. Clearly this problem cannot be done.*

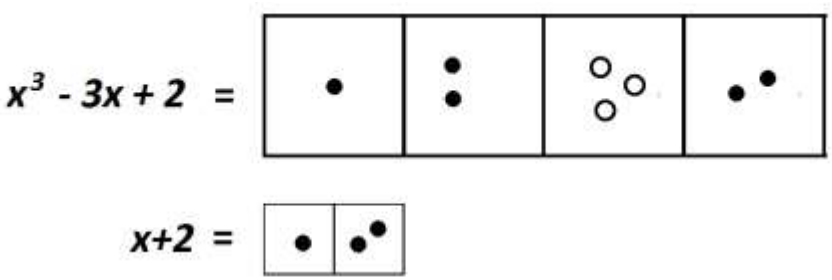
Comment: The audience knows we're playing a psychological game here, that there has got to be a way to make this work. And they are determined to make it work. This is the golden moment of the lecture. I just love this very moment!

Some knowledgeable person might yell out "add a zero pair" and, either I pretend I don't hear it, or I will decide that is the moment to carry on this way …

> *You know … here's why I loved being a high school teacher. The truth is that I don't really care tinkers about dividing polynomials. They are not important for everyday life – I've certainly never divided polynomials in my everyday living – and I really didn't care about them as a research mathematician. I don't think high-school students "need" to know about them at all!*
>
> *But I do love teaching them and think that what I am teaching *is* important. Polynomials, per se, I don't care one whit about that content material. But that content is vehicle for something very important. As a math teacher, I am teaching life lessons. And I really mean that, life lessons!*
>
> *To show you what I mean, here's a life lesson we have right now, at this very moment.*
>
> > *IF THERE IS SOMETHING IN LIFE YOU WANT – MAKE IT HAPPEN! (and deal with the consequences!)*
>
> *Right here. Is there anything in life we want right now? We're all looking at that single dot to the left. Is there anything you want to go with it?*

“Two dots next to it”

Then I say … MAKE IT HAPPEN! Let’s just put in two dots next to that single dot.

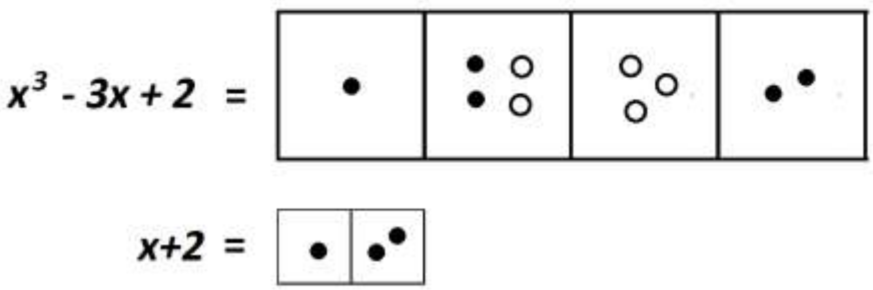


I keep my hand on that second box to subtly indicate that I haven’t finished with it.

Great! But we have some consequences. That box is meant to be empty after all.

“Put in two anti-dots”

Brilliant!



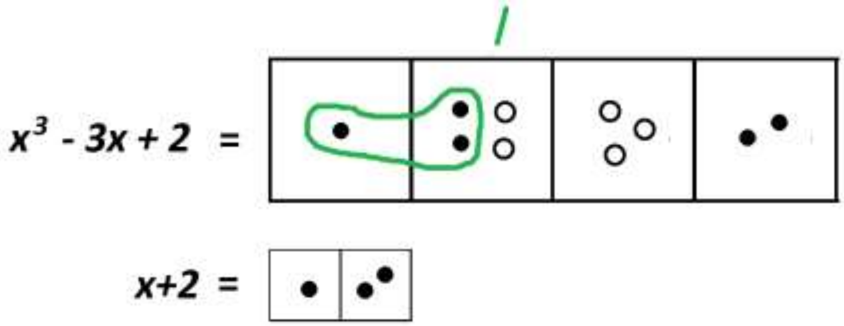
And is the box still technically empty?

“Yes.”

Comment: I might respond to the jargon “zero pair” if it seems appropriate, but always make the comment about how annoying special jargon is … just do it!

Okay. Next question. Was doing this helpful?

Well, we do at least see one copy of the thing we are looking for.

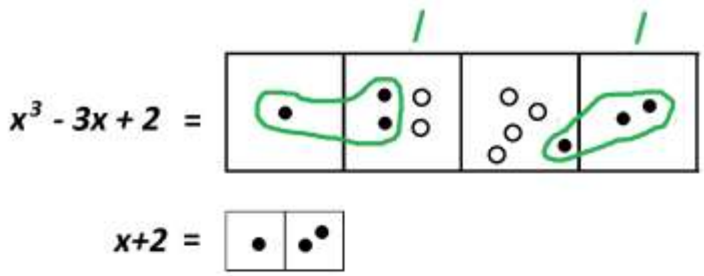


Hmm. Maybe that isn't very helpful. We can't seem to go any further.

"Do it again. Add another dot and antidot."

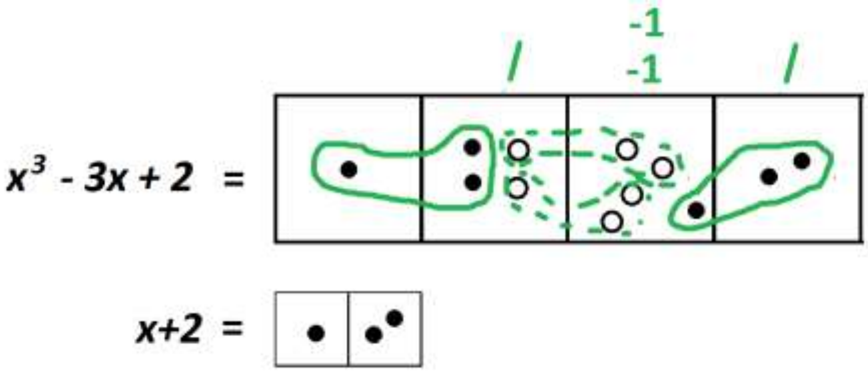
Oh! You are seeing the two dots to the far right. Is there anything in life you want right now to go with them? Yep. One dot to their left. And to deal with the consequences, we have to add an antidot as well. Okay. That gets us a little bit further.

But is all this helpful?



Now I usually stay silent, maybe adding a comment about wanting to weep again if the silence feels a bit awkward, until someone asks, usually timidly: "Don't we have an anti version of what we are looking for?"

Oh heavens! Yes! You are saying that this is the exact opposite of what we're looking for: one anti-dot next to two anti-dots. So we've got negative one of what we are looking for at this level, and another negative one of them.



WHOA! So how do we read this answer?

"$x^2 - 2x + 1$"

Brilliant!

So actually one can do all polynomial problems this way, even ones with negative coefficients.

Comment: This can be a satisfying place to stop if time is up or if it just feels right from the feel of the audience.

Do you want to do one on your ownsies?

Try: $\dfrac{4x^5 - 2x^4 + 7x^3 - 4x^2 + 6x - 1}{x^2 - x + 1}$. *If you can do this one you really can do any one!*

Warning: I suggest you draw really big boxes for this one. You'll need to add lots of dots and antidots.

This is a bit of a struggle as the picture can easily become messy. I do it on the board to see the answer $4x^3 + x^2 + 5x - 1$.

Comment: This too is a satisfying place to stop if time is up or if it just feels right from the feel of the audience.

Comment: If the audience wants more I do

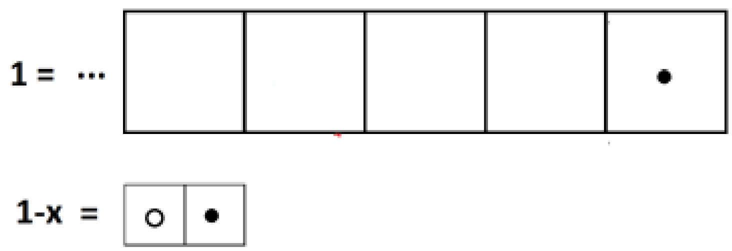
$$\frac{x^5-1}{x-1}$$

to get $x^4+x^3+x^2+x+1$ or perhaps $\frac{x^{10}-1}{x-1}$ if I am feeling cheeky.

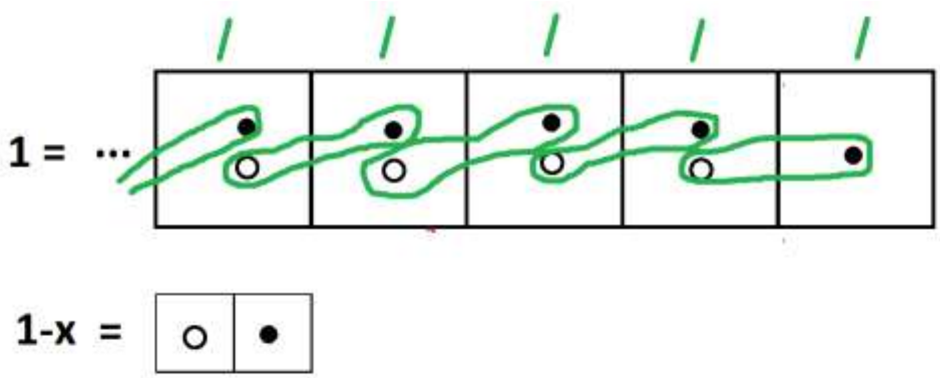
If the audience wants me to go to even more advanced algebra I'll invite them to do

$$\frac{1}{1-x}.$$

This is the polynomial 1 (a single dot) divided by a $1-x$, which looks like an anti-dot and a dot.



We see an infinite process.



I usually have to help the audience read this answer.

We have $\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$ *forever.*
Have you seen this formula before? In school books it is usually written the other way round:

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

Actually, school books usually use the letter r *here.*

$$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r} \ .$$

They sometimes have an a *thrown in for reasons I don't understand.*

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1-r}$$

Anyway, it is called the "geometric series formula."

Okay, I really am going to stop now. But I will leave you with a piece of optional homework. What does "optional" mean? "Don't do it under any circumstances."

Work out $\frac{1}{1-x-x^2}$.

You'll discover why number theorists like these polynomial divisions that give infinite sums. In this polynomial division problem you'll discover the Fibonacci numbers! Whoa!

Try it!

Comment: This is, for sure, a fine place to stop.
(I might occasionally do the work of the $2 \leftarrow 3$ machine to discover something one might want to call a version of base one-and-a-half, but it is better suited for a longer session.)