Assessment Thoughts: A few past thoughts relevant for today?

James Tanton

Last month I posted the following tweet.

> *If we value proplem-solving, deep thinking, grit, joyful exploration, creativity, and [the] confident play of ideas in math and wish this for all our Ss* (that's twitter for "students"), *then why do we test and grade on memorization, speed fluency, [and] short-term memory achievement?*

It was motivated by Chris Brownell's article Grades are for Onions, Beef, and other Produce: Not Children and Mathew Beyranevand's Retaking Assessments: Many Math Teachers are Late to the Party then followed up by Sunil Singh's Applying Archaic Math Assessment Ideas to Other Skills/Activities. These pieces reminded me of an essay I wrote whilst in the midst of being a full-time high-school teacher grappling with the very issues they each raise.

Testing dictates the culture of the classroom. It simply does and it will absolutely continue to do so if high-stakes assessment is used to "evaluate" educators' teaching effectiveness (and hence salary and job security) and students' perceived competency (and hence perceived adequacy for college). And I don't see that changing any time soon. The question is then: How can we take first steps—any steps—from within the system to make a shift away from the testing culture?

I share below my thoughts from eight to ten years ago, some baby steps that I personally practiced as a high-school teacher.

The essay is long and comes in these sections:

CONTENTS

A FEW THOUGHTS ABOUT ASSESSMENT

James Tanton

I was recently asked what my wish was for the next generation of students coming through our mathematics education system. I wrote:

> *[For each student] a personal sense of curiosity coupled with the confidence to wonder, explore, try, get it wrong, flail, go on tangents, make connections, be flummoxed, try, wait for epiphanies, lay groundwork for epiphanies, go down false leads, find moments of success nonetheless, savor the "ahas," revel in success, and yearn for more.*

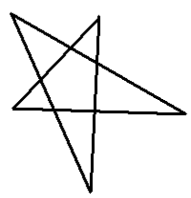
This might not be the answer one would expect, but it came from the heart without my thinking about it.

So much of the mathematics curriculum in past decades has been focused on skill and "what" questions:

- What is measure of $\angle ABD$?
- What is the percentage increase?
- What is 357×892?
- What is the equation of the tangent line?

As an educator for the past 11 years I have been working to find the wiggle room within the rigid system to ask with my students *why* and *what else* and *what if* human questions.

- Why should 357×892 and 892×357 give the same answer?
- Discover something interesting about the angles in a randomly drawn five pointed star. Anything to be discovered in six- and seven-pointed stars?



- Who chose the number "360" for the number of degrees in a circle and why that number?
- Really, why is negative times negative positive?
- Is it better to use my 30% discount first and then apply sales tax, or compute sales tax first and then apply the 30% discount?
- What's the easiest way to compute 15% of 6.4?
- Why are equations of the form $ax^2 + bx + c = 0$ called "quadratics"? Where is the number four?
- Why is dividing by zero not allowed?
- Why is multiplying by 10 the same as "adding a zero at the end"?
- Is the number e one learns when studying compound interest the same as the number e one learns when differentiating exponential functions?
- Why do logarithms convert multiplications into additions? Why was the scientific world so excited back in the 1600s when Napier want offered them?

There is no doubt that one needs to develop facility and ease with the basic skills of mathematics so that one is not always stumbling over small matters. But the richness and true utility of the subject comes from its conceptual structure – the problem solving and analytic tools it develops and the power of the intellectual penetration it promotes. (And as a mathematician, I would add the shocking beauty and poetry of mathematics offers its own value and inspiring reward!)

The Common Core State Standards in Mathematics are trying to bring mathematical thinking, over rote doing, into the classroom. The eight Practice Standards, at the very least, address this. Matters are directly shifting and there is encouragement to bring mindful reflection into the student mathematics experience.

But, in practice, the typical classroom is severely pressed by the weight of mandated tests and exams. ("There simply isn't time to play with ideas.") Teachers practice their art as best they can within the parameters given, and students, by and large, perform the exactly appropriate skills needed to survive and be rewarded by the system: don't question, memorize and just do.

So … *Where is the wiggle room?*

ARE WE CLEAR ON OUR GOALS?

What is the ultimate drive of the upper-school curriculum mathematics experience?

Is it that we regard the high-school content as vital for everyday life?
In which case, tell me when you last used the quadratic formula or complex numbers in your daily activities? Do we need to go beyond grade 7 or 8 for routine worldly function?

Is it the practical utility of the subject for advanced applications?
All state standards seem to suggest there is more to the curriculum content than needing to know basic matrix algebra, polar coordinates, and the like for engineering. (All students will likely be engineers?)

Is it to push towards calculus? Calculus is, after all, the first baby step needed for serious scientific work.
In which case, why do we not begin teaching calculus concepts early on? Why wait to the senior year? Students play with calculus concepts when they first examine the area of a circle, approximate the area of their handprints on graph paper, and play with infinite decimals and ask "Is 0.999…. equal to one or is it not?"

We all understand that there is indeed human value in what we teach at the upper-school levels. But many might say that the details of the content are not actually the focus – the thinking behind them is and the process one experiences in learning how to learn something complex is key.

Mathematical thinking promotes:

- Flexible and powerful thinking
- Assuredness through systematic processing
- Perspective and connection between disparate ideas
- Clarity
- The confidence to think one’s way through problems

That is, mathematical thinking promotes life skills! We could argue that the content of the curriculum is the vehicle for learning, not itself the goal of the learning.

At the upper-school levels we work on intellectual maturity, of growth, of depth of understanding and flexibility of thought, on learning how to ask questions, to create questions and to extend and push boundaries. These, after all, are the skills required for true success with innovation in business and breakthroughs in scientific research.

The Goal of Beauty:
There is no doubt that mathematics possesses immeasurable utility (indeed a considerable bulk of mathematics is inspired and motivated by “real world” practical problems). But the mathematics that comes of it soon begins to speak to something deeper. Mankind has not engaged in mathematics for thousands of years simply because it is useful, but because it speaks to something transcendental. We should actively work to share this aspect of mathematics too! Why play the violin? Because it is beautiful. Why do math? Because it is beautiful.

A Word on Words
Take note of the words we regularly use for things we make students do in the mathematics classroom.

exercise
problem
challenge
exam
worksheet

We discuss how we might go about *attacking* a problem.

And what is the standard extracurricular math activity? Joining a math team to prepare for *competition*.

How did this combative language come to be in mathematics?

Rather than “attack a problem” can we perhaps “probe an idea”? Rather than do thirty exercises for homework (and why not add thirty push-ups to that as well?) can we engage in explorations? Can the first words that come to mind in describing “extracurricular mathematics” be joyful intellectual play?

RULES versus TOOLS (A phrase coined by Anne Watson)

I have a sign over my classroom whiteboard which reads

> *My ultimate goal as a teacher is to transform procedural understanding into conceptual understanding. (Conceptual understanding is so much more powerful and much more fun!)*

It is my statement to students that intellectual play focused towards conceptual understanding is the route to true success and meaning. Getting the right answer, though important, is in some sense secondary.

This view of matters is often very unsettling to students. Many have been taught, and rewarded, to focus on getting the right answer—and quickly. Textbooks give the impression that this is the goal.

> *Answer these questions that you yourself have not asked and get the same answers listed at the back. Don't worry – these are not new questions – someone else has answered them all already. Be sure to do at least 40 of them for homework tonight. (And while you are at it, do some push-ups.)*

In order to succeed with the "get the right answer quickly" approach students often do that which is utterly appropriate for that goal: Memorize rules that lead to the right answers.

- To multiply by 10 add a zero.
- To divide by a fraction, flip and multiply.
- The vertex is at $x = -b/2a$.
- $|x-a| > b$ is OR; $|x-a| < b$ is AND.
- The derivative of a^x is $\ln a \cdot a^x$.
- FOIL.

There is now doubt that mathematicians use these quick rules too—when I multiply by 10, I do indeed just "add a zero at the end" –but mathematicians work from a base of understanding first. I know, for example, that when working in base two multiplying by 2 just "adds a zero at the end" too. (Is that obvious?)

So an alternative version of my classroom whiteboard statement would be: *Let's turn rules into tools.*

ON ROUTINE ASSESSMENT

Don't get me wrong: Routine exercises and practice problem sets on skills and procedures have their place. One must practice ideas and make lots of small errors (and catch them!) when learning a sequence of new ideas. Traditional homework assignments, graded pieces, quizzes and tests are an integral part of the school room experience. But one can ask …

On the Structure of Tests:

- Do quizzes and tests need to be timed? To what extent does it matter how long a student takes to complete a set of problems?
- Can some quizzes be done in pairs?
- Can some quizzes or exams be open-note exams?
- How about oral tests?
- Can students grade each others' work?
- Can students be asked to submit possible questions for an upcoming exam? (Teaching the art of writing good questions is worthwhile pursuit!)
- Why give numerical scores in math? Why not give letter grades based on thinking, clarity of expression, and so on?

I am sure you can think of many more possibilities to add to this list.

On Homework: *Let go of busy work for the sake of busy work.*

> If handed a long list of homework problems to give to your students, why not turn that into an opportunity for students to practice and assess their own learning approaches? Give these instructions:
>
> *1. Answer enough questions from this list to reach the point you feel you understand what is going on.*
>
> *2. Of the questions you left unsolved, pick the one you feel will be the easiest to do and the one you feel will be the hardest to solve. Do them, and see if your sense of them were more or less right. Write me a sentence or two about this experience. (Be sure to let me know which two problems you worked on.)*
>
> *3. Of the harder questions you did, what made the hardest ones so hard? Make up one problem similar to these hard questions to give to a friend, one that uses the same ideas but is actually a bit easier to solve. Share that problem with me.*

In her essay "Adolescent Learning and Secondary Mathematics" (Proceedings of the 2008 Annual Meeting of the Canadian Mathematics Education Study Group/ Groupe canadien d'etude en didactique des mathematiques, 21-29), Anne Watson offers similar and further ideas of this type.

Alternative Graded Tests
I routinely give two types of quizzes that often raises an eyebrow among colleagues.

A 100% Packet is a large, involved problem set students complete over a period of weeks. Students must earn a perfect score of 100%! They may use their notes, they may consult with me for guidance, and they are expected to hand in the packet multiple times for partial grading and feedback. They must keep working at all the problems until they have a perfect score - otherwise it is a score of 0%. (I never ended up giving out a zero score by the way!)

A Quiz with all the Answers Supplied is a quiz with, well, all the answers supplied! *A chord in a circle of radius 10 subtends an arc of 42 degrees. What is the length of the chord? [The answer is 24.3.]* Doing this reinforces the point that I am not so much interested in the numerical answer but the method towards obtaining it. (There is also the psychological comfort of knowing whether or not your answer is correct.)

META-ASSESSMENT: Changing the Questions We Ask

To promote conceptual understanding we can ask *meta-questions* – questions about questions – and questions that force students to step outside of the question. Here are eight categories of such things. (I am sure you can come up with more!) I supply concrete examples to illustrate what I mean by each of them.

Be gentle if you start giving questions like these to your students: they can be a bit of a shock. They are outside of the familiar rote-skills approach to learning and doing.

1: SPOT THE ERROR

Question: In answering the question: XXXXXXX Lizzy wrote: YYYYYYYYY.

Why did the teacher give her a score of only 3/5 for her response?

Question:

a) A student writes in his homework: $6x = 18 = x = 3$. What does this actually say? What do you think the student was trying to say?

b) Another student writes in her homework: $\frac{3(x+2)}{5} = \frac{3x+6}{5} = \frac{3}{5}x + \frac{6}{5}$.

Is this saying something reasonable?

c) A student writes in his homework:

$$2(x-2) = 1 + x$$
$$= 2x - 4 = 1 + x$$
$$= x = 5$$

Do you think this is saying something reasonable? Any advice for the student?

Question: Consider the following geometry proof question: XXXXXXX. Gerald wrote: YYYYYYY. But there is a mistake in Gerald's work that causes his proof to logically collapse! What is Gerald's error?

Can you write a logically sound proof for this question?

Question: A student was asked to expand $x(x-2)$ and wrote $x = 0,\ 2$. What do you think the student was doing? Any advice?

2: CLASSIC ERRORS HEAD-ON

Battling with how hard it is to force something that is generally false to work cements the idea that it is generally false!

Question: Many students write:

$$(a+b)^2 = a^2 + b^2 .$$

a) Choose some specific values for a and b to show that this is not true in general.

b) Find a value for a and a value for b for which, by luck, $(a+b)^2 = a^2 + b^2$ happens to hold.

c) Use algebra to find <u>all</u> values for a and b for which $(a+b)^2 = a^2 + b^2$ happens to hold.

One can apply this type of question to many of the classic algebra errors.

$$\sqrt{a^2+b^2} = a+b \qquad \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} \qquad 3 \cdot \frac{a}{b} = \frac{3a}{3b} \qquad \frac{2x+y}{2z} = \frac{x+y}{z}$$

Question: The height of a star at an angle of elevation 30° is the same as the height of a start at 150°. Thus the following is true:

$$\sin\left(30^\circ\right) = \sin\left(150^\circ\right)$$

Lulu says: *Divide both sides by sin and get* $30 = 150$*.*

$$\cancel{\sin}\left(30^\circ\right) = \cancel{\sin}\left(150^\circ\right)$$
$$30 = 150$$

But clearly 30 does not equal 150!

How would you help Lulu out here? What concept is she struggling with do you think?

Question: Come up with a scenario for which "$\frac{1}{2}$ plus $\frac{1}{3}$" could legitimately have the answer $\frac{2}{5}$.

Question: In answering the question:

Find the equation of the tangent line to the curve $y = x^3 + 5x^2$ *at the point* $(1,6)$.

Poindexter wrote: $y - 6 = 3x^2 + 10x(x-1)$

What do you think of his answer?

Question: Gordie thinks that the following is a valid log rule

$$\log M \cdot \log N = \log(M + N).$$

He says that the equation is showing how multiplication is being turned into addition (which is indeed what logarithms do).

How would you advise Gordie on rethinking his approach to logarithms? How could you help him be clear in his mind about the way logarithms work?

3: EXPLAIN

Question: Joyce says that finding the value of $5^{\log_5 37}$ is manageable and swift if you think about it for a while. And she is right!

What is the value of this quantity and how would you explain to Quentin, who doesn't "get it," why the answer is what it is?

Question: Nervous Nelly, who prefers to memorize "rules" for mathematics, was once told that multiplying an inequality by a negative number "flips" the inequality.

For example, if $C < D$ then $-C > -D$. And if $x < -3$, then $-2x > 6$.

a) Has she memorized a correct rule?
b) Nelly admits she does not understand why the rule she memorized is true. How would you explain its validity to her?

Question: a) Work out $\frac{12}{15} \div \frac{3}{5}$ and show that it equals $\frac{4}{3}$.

b) Now notice that

$$12 \div 3 = 4$$
$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3}$$

Is this a coincidence? Or does $\frac{a}{b} \div \frac{c}{d}$ always equal $\frac{a \div c}{b \div d}$?

4: THINK BEFORE YOU LEAP!

Question: For each of the following describe an easy way to compute the answer without a calculator. Either describe your method in words or write a line of arithmetic that illustrates your way of proceeding.

a) 82×5 b) $35 \times 35 \times 40$ c) 7×16 d) 198×32

e) $87 \cdot 903 + 13 \cdot 903 + 17$ f) $196 - 37$ g) $817 - 69$

h) 621 divided by 5 i) 15% of 62 j) $\frac{13}{66} \cdot \frac{33}{28} \cdot \frac{7}{13}$

k) $603 \div 97$ l) $813 \div 198$

Question:
Here are four quadratic equations:

(A) $y = 4(x-3)(x-7)$

(B) $y = 3(x-2)^2 + 6$

(C) $y = 2x^2 - 4x + 8$

(D) $y = x^2 + x(x-3)$

i) For which equation would it be easiest to answer the question: *What is the vertex of the quadratic?*

ii) For which equation would it be easiest to answer the question: *Where does the quadratic cross the x-axis?*

iii) For which equation would it be easiest to answer the question: *What is the smallest value the quadratic adopts?*

iv) For which equation would it be easiest to answer the question: *What is the line of symmetry of the quadratic?*

v) For which equation would it be easiest to answer the question: *What is the y-intercept of the quadratic?*

Question: Which of the following problems is <u>not</u> easy to work out in your head?

$$23 \times 37 - 13 \times 37$$
$$27 \cdot 153 + 73 \cdot 153$$
$$3(7) + 87(7)$$
$$105(105) - 95(105)$$
$$17 \times 13 + 13 \times 3$$
$$34 \times 7 + 34 \times 6$$

Question: Compute: $(7-18)(8-18)(9-18)(10-18)\cdot\cdots\cdot(25-18)(26-18)$.

Question: Which of the following statements seem they could be true? Which are definitely wrong? (Answer this question without actually computing the products. Not one of them is actually correct! This is an exercise in estimation only.)

$$999 \times 31 = 30999$$
$$12 \times 198 = 1996$$
$$106 \times 213 = 206,816$$
$$9458 \times 9786 = 192837261748$$
$$19990 \times 4 = 76987$$

Question:

a) How big an answer do you expect from computing 19×998?
b) Would computing 5671×1772 give an answer in the millions?
c) Would computing 1123×1005 give an answer in the millions?
d) Would 997×998 give an answer in the millions?

Question: Quickly … solve:

a) $(x-2)(x-14)(x-22) = 0$
b) $(x+1)^2 = 25$

Question: A parabola passes through the points $(2,5)$, $(3,-6)$ and $(10,5)$. What is the x-coordinate of its vertex?

Question: Find the area of a triangle with side lengths 9 inches, 8 inches and 19 inches.

Question: Find $\lim_{h \to 0} \dfrac{(3+h)^2 - 81}{h}$.

Question: Evaluate:

$$\frac{\dfrac{4-2}{8-3}-2\cdot\dfrac{7-(12-6)}{30-\dfrac{5\times 5}{4-3}}}{\dfrac{19}{3}+\dfrac{17}{2+\dfrac{1}{\left(17-\dfrac{4}{5}\right)}}}$$

Question: Here are four problems. Don't answer them(!), but tell me what type of problem each is. Is it about

ADDITION AND SUBTRACTION of fractions

Or

MULTIPLICATION of fractions

Or

DIVISION of fractions?

(A) Tom's driveway is ¼ of a mile long. If Tom walks at a speed of three-and-a-half miles per hour, how long will it take him to walk the length of his driveway?

TYPE =

(B) One third of a field is planted with corn, one quarter with cabbages, and the rest with squash. What fraction of the field is planted with squash?

TYPE =

(C) John earns $3500. He gives 30% of income to the IRS and the one third of what remains to his mother. How much money does John have remaining?

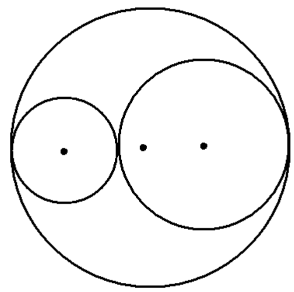
TYPE =

(D) Alfred purchases a suit, normally priced at $315 but was on sale for a 20% discount. If the state charges 8% sales tax, how much did Alfred end up paying for his suit?

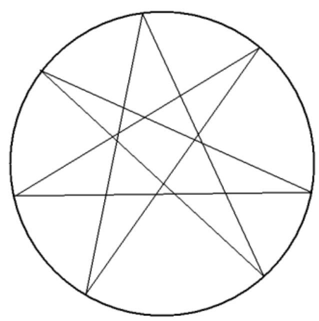
TYPE =

5: DISCOVER AND EXPLORE

Question: The centers of all the circles in this picture are collinear. Discover and explain something interesting about the circumferences of these circles



Question: Discover and explain something interesting about the angles of a pointed star drawn inside a circle as shown.



Question: Playing on a calculator Pandi noticed that 2^{46}, 2^{56} and 2^{76} each begin with a seven.

a) Find the next few powers of two that begin with a seven.
b) Explain why the pattern you are seeing stops.

Question: Sketch $y = x^{\frac{1}{\ln x}}$ on a calculator. Explain!

6: JOLT!

Bring in unexpected connections to provoke clarity!

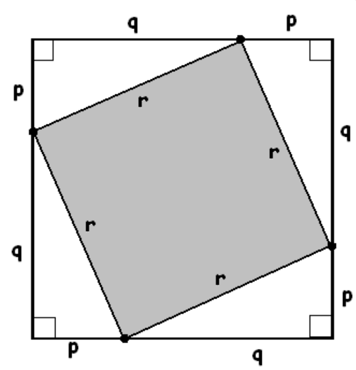
Question: a) Compute $(x^6 + x^5 + 5x^4 + 5x^3 + 9x^2 + 5x + 2) \div (x^2 + x + 2)$

b) Put $x = 10$ into the problem of part a). What grade-five division problem have you just solved? Does your answer seem to be correct?

People forget in an algebra course that x can actually be a number!

Question:
a) Show that $x^5 - 1$ is divisible by $x - 1$.
b) Is $2^{100} - 1$ prime?

Question: Compute the area of the central shaded square two different ways:



Question: Consider the differential equation $\frac{dy}{dx} = iy$ with $y(0) = 1$.

a) What would be the standard solution to this differential equation?
b) Show that $y = \cos x + i \sin x$ is also a solution.

What might you be tempted to now say?

7: PUSH BOUNDARIES

Question:
In computing 654 + 179 Iggy writes:

$$\begin{array}{r} 654 \\ +\,179 \\ \hline \end{array}$$

$$7\ \ 12\ \ 13 \quad = 833$$

Does this represent valid mathematical thinking? Briefly explain what you guess Iggy was thinking.

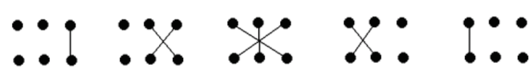
Question:

a) When asked to find a fraction between $\frac{1}{12}$ and $\frac{1}{11}$, Poindexter wrote: $\frac{1}{11\frac{1}{2}}$.

Is this a mathematically valid answer?

b) Quickly write down 99 fractions that lie between $\frac{1}{12}$ and $\frac{1}{11}$.

Question: Vedic mathematics taught in India (and established in 1911 by Jagadguru Swami Bharati Krishna Tirthaji Maharaj) has students compute the product of two three-digit numbers as follows:



What do you think this sequence of diagrams means?

Question: a) Explain why the following party trick works:

Think of two single-digit numbers.
Take the first digit and multiply it by 5.
Add 5 and then multiply the result by 4.
Add the second number you thought of and subtract 20.
Add the second number you thought of a second time, and then halve the result.
You are now thinking of a two digit answer.
Your answer is composed of the two digits you first thought of!

b) Make up your own party trick that involves thinking of three single digit numbers to produce a result that is a three-digit number composed of those three initial digits.

8: SOMETHING IS NOT RIGHT

These are like "spot the error" questions but require a deeper level of thought.

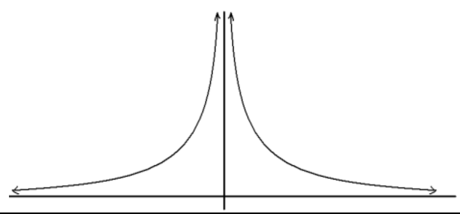
Question:
Could $56452 \times 18863 = 98611987364$ be correct?

Question: My calculator says that $\sqrt{46}$ equals 6.782329983. Why can't this be correct?

Question: Could the sum of 19000 odd numbers end with a five?

Question: Explain why the formula $(a+b)^4 = a^4 + b^4 + 4ab^3 + 6ba^3 + 4a^2b^2$ cannot be right.

Question: When I plot the curve $y = \dfrac{200}{1+x^4}$ on my calculator I see a vertical asymptote at $x = 0$. Do you? Should you?



Question: Sketch the graphs $y = x^2$ and $y = 2^x$ simultaneously on a calculator to se they intersect twice. Is this right? Do they intersect exactly two times?

Question: Tatiana says $10! = 8542677640$.
Quickly, why must she be mistaken?